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Stability of Coupled Oscillators using Frenkel-Kontorova Model

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***Abstract*— We study the stability of coupled oscillators motivated by the Frenkel-Kontorova (FK) model. The FK model describes a chain of classical particles coupled to their neighbors and subject to a periodic on-site potential. The open-loop system of the FK model represents interconnected oscillators that have locally stable or unstable equilibrium points. We reveal the stability of the coupled system in the presence of linear and nonlinear particle interactions, respectively, and verify the results by numerical simulations. The result applies to physical systems such as atomic-scale friction whose dynamics is described by the FK model.**

I. INTRODUCTION

The research in collective motion has progressed rapidly from early modeling and simulation of specific examples ([1], [2]) towards a more fundamental explanation and control applicable to a wide range of systems with collective behaviors ([3], [4], [5], [6]). In physics, the phenomenon of collective synchronization, in which coupled oscillators lock to a common frequency, was studied in the early work [7], [8]. After Kuramoto proposed the famous Kuramoto model for oscillator synchronization [9] in the 1970s, a related problem, the collective motion and phase transition of particle systems, is considered from the perspective of analogies to biologically motivated interactions [10], [11]. Despite thirty years having elapsed since Kuramoto proposed his important model, there remain important theoretical aspects of the collective motion that are not yet understood [12], which stimulates a large amount of research activities in controls community, see [13], [14], [15], [16] and references therein.

In this paper, we study stability of coupled oscillators which is motivated by the so-called Frenkel-Kontorova (FK) model ([17]). The FK model describes a chain of classical particles coupled to their neighbors and subject to a periodic on-site potential, see Figure 1. It characterizes the fundamental physics in problems such as sliding of nanoparticle array [18], [19], DNA dynamics [20], charge-density waves, magnetic spirals, and absorbed monolayers [21]. Recently, the FK model is used to describe frictional dynamics of a one-dimensional particle array that slides on a substrate [22], [23]. We consider the open-loop stability problem for the FK model, and study single particle stability in the coupled particle array subjected to a periodical substrate potential in the paper. The particle interactions are assumed to have either a linear or a nonlinear Morse-type form. We perform MATLAB simulations to verify the theoretical results.

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II. THE FRENKEL-KONTOROVA MODEL

The basic equations for the driven dynamics of a one dimensional particle array of N identical particles moving on a surface are given by a set of coupled nonlinear equations[21]:

$$m'\ddot{z}_i + \gamma'\dot{z}_i = -\frac{\partial U(z_i)}{\partial z_i} - \sum_{\substack{j=1 \\ j\neq i}}^{N} \frac{\partial W(z_i - z_j)}{\partial z_i} + f_i' + \eta, \tag{1}$$

where $i = 1, \dots, N$, z_i is the coordinate of the ith particle, m' is its mass, γ' is the positive friction coefficient representing the single particle energy exchange with the substrate, f_i' is the applied external force, $\eta(t)$ denotes additive the Gaussian noise, $U(z_i)$ is the periodic potential applied by the substrate, and $W(z_i - z_j)$ is the inter-particle interaction potential.

Under the assumptions that 1) the substrate potential is periodic, 2) the same force is applied to each particle, and 3) there is zero noise, the equation of motion reduces to the following simplified FK model:

$$\ddot{\phi}_i + \gamma\dot{\phi}_i + \sin(\phi_i) = f + F_i(\phi_i, \phi_j) \tag{2}$$

where ϕ_i is the dimensionless phase variable, $\phi_i = 2\pi z_i$, $F_i(\phi_i, \phi_j), j = i \pm 1$, is the Morse-type nearest-neighbor interaction ([18], [24]):

$$\begin{aligned} F_i(\phi_i, \phi_j) &= \frac{\kappa}{\beta}\left\{e^{-\beta(\phi_{i+1}-\phi_i)} - e^{-2\beta(\phi_{i+1}-\phi_i)}\right\} \\ &\quad -\frac{\kappa}{\beta}\left\{e^{-\beta(\phi_i-\phi_{i-1})} - e^{-2\beta(\phi_i-\phi_{i-1})}\right\}, \\ &\quad i = 2, \dots, N-1, \end{aligned} \tag{3}$$

where κ and β are positive constants, and the free-end boundary conditions are represented as:

$$\begin{aligned} F_1(\phi_1, \phi_2) &= \frac{\kappa}{\beta}\left\{e^{-\beta(\phi_2-\phi_1)} - e^{-2\beta(\phi_2-\phi_1)}\right\}, \\ F_N(\phi_{N-1}, \phi_N) &= -\frac{\kappa}{\beta}\left\{e^{-\beta(\phi_N-\phi_{N-1})} - e^{-2\beta(\phi_N-\phi_{N-1})}\right\}. \end{aligned}$$

As $\beta \to 0$, (3) turns to:

$$F_i(\phi_i, \phi_j) = \kappa\left(\phi_{i+1} - 2\phi_i + \phi_{i-1}\right), i = 2, \dots, N-1, \tag{4}$$

which represents a linear approximation of particle interaction for small β with the following free-end boundary

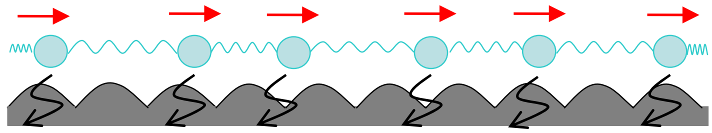

Fig. 1. The Frenkel-Kontorova model represents a harmonic chain (which mimics a layer of nano-particles) in a spatially periodic potential (which mimics the substrate). The chain is driven by a constant force which is damped by a velocity-proportional damping.

conditions:

$$\begin{aligned} F_1(\phi_1,\phi_2) &= \kappa(\phi_2-\phi_1), \\ F_N(\phi_{N-1},\phi_N) &= \kappa(\phi_{N-1}-\phi_N). \end{aligned} \tag{5}$$

The FK model presents a nonlinear interconnected system. The nonlinear Morse-type interaction represents an attraction force between two nearest particles when their distance is longer than the natural length of the spring, and a restoring force (increasing unlimited) between them when the distance is shorter than the natural length of the spring. Without the external force, *i.e.*, $f=0$, (2) represents coupled-oscillators whose individual dynamics is the second-order pendulum equation ([25]). We study the open-loop stability of (2) in the presence of linear and nonlinear Morse-type particle interactions in the next two sections, respectively.

III. STABILITY WITH LINEAR PARTICLE INTERACTIONS

The dynamics in (2) expressed without external forces can be expressed in state space as:

$$\begin{aligned} \dot{x}_{i1} &= x_{i2} \\ \dot{x}_{i2} &= -\sin x_{i1} - \gamma x_{i2} + F_i(x_{i1},x_{j1}), \end{aligned} \tag{6}$$

where $i=1,2\ldots,N$, $x_{i1}=\phi_i, x_{i2}=\dot{\phi}_i$, $F_i(x_{i1},x_{j1})$ takes the form of the linear nearest-neighbor interaction given in (4), that is,

$$F_i(x_{i1},x_{j1}) = \kappa\left(x_{i+1,1}-2x_{i1}+x_{i-1,1}\right). \tag{7}$$

From (6), the equilibrium points are at $(x_{i1},x_{i2})=(x_{i1}^*,0)$ where x_{i1}^* are solutions to

$$\begin{aligned} -\sin x_{11}^* + \kappa(x_{21}^*-x_{11}^*) &= 0, \\ -\sin x_{i1}^* + \kappa(x_{i+1,1}^*-2x_{i1}^*+x_{i-1,1}^*) &= 0, \\ i=2,\ldots,N-1,& \\ -\sin x_{N1}^* + \kappa(x_{N-1,1}^*-x_{N1}^*) &= 0. \end{aligned} \tag{8}$$

Linearize (6) around its equilibrium $(x_{i1}^*,0)$, and define new states $z_{i1}=x_{i1}-x_{i1}^*, z_{i2}=x_{i2}$. We obtain

$$\begin{aligned} \dot{z}_{i1} &= z_{i2} \\ \dot{z}_{i2} &= -\cos x_{i1}^* z_{i1} - \gamma z_{i2} + \kappa(z_{i+1,1}-2z_{i1}+z_{i-1,1}) \\ &\quad -\sin x_{i1}^* + \kappa(x_{i+1,1}^*-2x_{i1}^*+x_{i-1,1}^*) \\ &= -\cos x_{i1}^* z_{i1} - \gamma z_{i2} + \kappa(z_{i+1,1}-2z_{i1}+z_{i-1,1}) \end{aligned} \tag{9}$$

Note that the last equal sign holds due to the equilibrium equation (8).

Stacking the state space equations for $i=1,2,\ldots,N$, we obtain

$$\begin{aligned} \dot{z} &= diag\{A_i\}z + (I_N\otimes B)(Q_1\otimes\begin{bmatrix}\kappa & 0\end{bmatrix})z \\ &= diag\{A_i\}z + Q_1\otimes\begin{bmatrix}0 & 0\\ \kappa & 0\end{bmatrix}z \end{aligned} \tag{10}$$

where $z=[z_{11},\ z_{12},\ z_{21}\ z_{22},\ \ldots\ z_{N1},\ z_{N2}]^T$,

$$A_i=\begin{bmatrix}0 & 1\\ -\cos x_{i1}^* & -\gamma\end{bmatrix},\qquad B=\begin{bmatrix}0\\1\end{bmatrix}, \tag{11}$$

$$Q_1 = \begin{bmatrix} -1 & 1 & 0 & \ldots & 0 \\ 1 & -2 & 1 & 0 & \ldots \\ & & \vdots & & \\ 0 & \ldots & 1 & -2 & 1 \\ 0 & \ldots & 0 & 1 & -1 \end{bmatrix} \in \Re^{N\times N}, \tag{12}$$

Kronecker product and its property are reviewed in Appendix A.

Notice that (10) represents the linearized coupled oscillator equation, where the term, $diag\{A_i\}z$, represents the linearized pendulum equation and Q_1 represents the nearest-neighbor interconnection. Notice also that $(-Q_1)$ is a row-sum-zero matrix whose property is given in Lemma 3 of Appendix.

To reveal stability of (10), we re-write it into a different but equivalent matrix representation:

$$\dot{z} = (I_N\otimes A)z + Q_2\otimes\begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix}z \tag{13}$$

where

$$A=\begin{bmatrix}0 & 1\\ 0 & -\gamma\end{bmatrix}, \tag{14}$$

and Q_2 is represented in (15).

We have the following theorem:

Theorem 1: The system (6) with linear particle interaction (4) is locally asymptotically stable at the equilibrium points $(x_{i1}^*,0)$ if all of the eigenvalues of the matrix Q_2 defined in (15) have negative real parts; it is unstable if any of the eigenvalues of the matrix Q_2 has a positive real part. Particularly, it is locally asymptotically stable if $\cos x_{i1}^*\geq 0$ for all i with strict inequality for at least one i, and it is unstable if $\cos x_{i1}^*\leq 0$ for all i with strict inequality for at least one i.

Proof of Theorem 1: First, we study stability of the linearized system (13) for any positive constants γ,κ and for any $N\geq 2$. To perform this stability analysis, we find a

$$
Q_2 = \begin{bmatrix} -\kappa - \cos x_{11}^* & \kappa & 0 & \dots & 0 \\ \kappa & -2\kappa - \cos x_{21}^* & \kappa & 0 & \dots \\ & & \vdots & & \\ 0 & \dots & \kappa & -2\kappa - \cos x_{N-1,1}^* & \kappa \\ 0 & \dots & 0 & \kappa & -\kappa - \cos x_{N1}^* \end{bmatrix} \in \Re^{N\times N}. \quad (15)
$$

transformation matrix to transform the system matrix into a block diagonal one.

Define a similarity transformation $z = \overline{T}\zeta$. In the new coordinate, the system dynamics is

$$
\dot{\zeta} = H\zeta. \quad (16)
$$

We show how to choose $\overline{T}$ and present H accordingly.

Since Q_2 is a real symmetric matrix, according to Lemma 1, there exists a unitary matrix T such that $T^{-1}Q_2T = D$ where D is a diagonal matrix of eigenvalues of Q_2. Let

$$
\overline{T} = T \otimes I_2 \quad (17)
$$

where I_2 is the 2×2 identity matrix. Then:

$$
\begin{aligned}
H &= \overline{T}^{-1}\left(I_N \otimes A + Q_2 \otimes \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right)\overline{T} \\
&= \left(T^{-1}I_NT\right) \otimes A + \left(T^{-1}Q_2T\right) \otimes \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \\
&= I_N \otimes A + D \otimes \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \quad (18)
\end{aligned}
$$

We can see that H is block diagonal, and the block diagonal element of H writes:

$$
H_{ii} = \begin{bmatrix} 0 & 1 \\ \alpha_i & -\gamma \end{bmatrix}, \quad (19)
$$

where $\alpha_i, i = 1, 2, \dots, N$ are eigenvalues of Q_2. The stability of the system depends on the sign of the real parts of $\alpha_i, i = 1, \dots, N$:

1) If $\alpha_i, i = 1, 2, \dots, N$ have negative real parts, the eigenvalues of $H_{ii}, i = 1, 2, \dots, N$ have also negative real parts, and so does the matrix H. This indicates that the system is asymptotically stable at these points. Due to the similarity transformation, the same stability result holds for the original system (13). Furthermore, local stability of the original nonlinear system (6) can be deduced from the stability analysis of its linearized system (13) ([26], Theorem 3.1).
2) If α_i has a positive real part for any $i \in [1, N]$, eigenvalues of $H_{ii}, i = 1, 2, \dots, N$, also have positive real parts. With the same arguments as above, the system (6) is unstable at these points.

Checking the structure of matrix Q_2 in (15), we have the following cases:

- If $\cos x_{i1}^* \geq 0$ for all i with strict inequality for at least one i, the matrix $(-Q_2)$ is an M-matrix and $\alpha_i < 0$ for all i according to Lemma 2. Therefore, Q_2 is Hurwitz and the system is asymptotically stable;
- If $\cos x_{i1}^* = 0$ for all i, Q_2 has one (and only one) eigenvalue 0 according to Lemma 3. The linear system (13) is marginally stable and the stability of the nonlinear system (6) could be either stable or unstable;
- If $\cos x_{i1}^* \leq 0$ for all i with strict inequality for at least one i, we can represent Q_2 as in (20). Since Φ is a irreducible and nonnegative matrix, it has a positive eigenvalue, r, equal to the spectral radius of Φ, which is between $2k + \min\{-\cos x_{11}^*, \dots, -\cos x_{N1}^*\}$ and $2k + \max\{-\cos x_{11}^*, \dots, -\cos x_{N1}^*\}$ ([27], page 537). Therefore, Q_2 has at least one positive eigenvalue. The system is unstable;
- If $\cos x_{i1}^*, i = 1, \dots, N$, have mixed signs, the system could be either stable or unstable and numerical calculations is necessary to determine the sign of the real parts of the eigenvalues of Q_2.

This concludes the proof of Theorem 1. $\square$

Remark 1: From the proof of Theorem 1, we can see that for a given set of equilibrium points, 1) if at least one of the individual oscillators is stable and the rest is marginally stable, the interconnected oscillator is locally asymptotically stable; 2) if at least one of the individual oscillators is unstable and the rest is marginally stable, the interconnected oscillator is locally unstable; 3) if some of the individual oscillators are stable and some unstable, the stability of the interconnected oscillator may be locally stable or unstable.

Remark 2: As special cases of Theorem 1, the equilibrium points $(2k\pi, 0), k = 0, \pm 1, \dots$, are asymptotically stable and $((2k+1)\pi, 0)$ are unstable. The result was first claimed in our early publication [19] without rigorous proof, and then in [28] with a proof. We extend the result in this work to include all equilibrium points of the coupled oscillators.

IV. STABILITY WITH NONLINEAR PARTICLE INTERACTIONS

In the presence of Morse-type nonlinear particle interactions, that is, F_i takes the form (3), the equilibrium points of (6) are at $(x_{i1}, x_{i2}) = (x_{i1}^*, 0)$ where x_{i1}^* are solutions to

$$
\begin{aligned}
&-\sin x_{11}^* + \frac{\kappa}{\beta}\left\{e^{-\beta(x_{21}^* - x_{11}^*)} - e^{-2\beta(x_{21}^* - x_{11}^*)}\right\} = 0, \\
&-\sin x_{i1}^* + \frac{\kappa}{\beta}\left\{e^{-\beta(x_{i+1,1}^* - x_{i1}^*)} - e^{-2\beta(x_{i+1,1}^* - x_{i1}^*)}\right\} \\
&\quad -\frac{\kappa}{\beta}\left\{e^{-\beta(x_{i1}^* - x_{i-1,1}^*)} - e^{-2\beta(x_{i1}^* - x_{i-1,1}^*)}\right\} = 0, \\
&\qquad i = 2, \dots, N-1, \\
&-\sin x_{N1}^* \\
&\quad -\frac{\kappa}{\beta}\left\{e^{-\beta(x_{N1}^* - x_{N-1,1}^*)} - e^{-2\beta(x_{N1}^* - x_{N-1,1}^*)}\right\} = 0
\end{aligned}
$$

$$Q_2 = \begin{bmatrix} \kappa - \cos x_{11}^* & \kappa & 0 & \dots & 0 \\ \kappa & -\cos x_{21}^* & \kappa & 0 & \dots \\ & & \vdots & & \\ 0 & \dots & \kappa & -\cos x_{N-1,1}^* & \kappa \\ 0 & \dots & 0 & \kappa & \kappa - \cos x_{N1}^* \end{bmatrix} + (-2\kappa) I_N \stackrel{def}{=} \Phi + (-2\kappa) I_N. \quad (20)$$

(21)

Let $z_{i1} = x_{i1} - x_{i1}^*, z_{i2} = x_{i2}$, and linearize the system around its equilibrium $(x_{i1}^*, 0)$. After simplification, we get

$$\begin{aligned} \dot{z}_{i1} &= z_{i2} \\ \dot{z}_{i2} &= -\cos x_{i1}^* z_{i1} - \gamma z_{i2} \\ &\quad + \frac{\kappa}{\beta}\left[-e^{-\beta(x_{i+1,1}^* - x_{i1}^*)} + 2e^{-2\beta(x_{i+1,1}^* - x_{i1}^*)}\right] \\ &\quad (z_{i+1,1} - z_{i1}) \\ &\quad - \frac{\kappa}{\beta}\left[-e^{-\beta(x_{i1}^* - x_{i-1,1}^*)} + 2e^{-2\beta(x_{i1}^* - x_{i-1,1}^*)}\right] \\ &\quad (z_{i1} - z_{i-1,1}) \\ &\stackrel{def}{=} -\cos x_{i1}^* z_{i1} - \gamma z_{i2} + c_{i1}(z_{i+1,1} - z_{i1}) \\ &\qquad -c_{i2}(z_{i1} - z_{i-1,1}) \end{aligned} \quad (22)$$

We can see that (22) is in the same form as in (9) with different coupling coefficients. We can represent (22) as

$$\dot{z} = (I_N \otimes A)z + Q_3 \otimes \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} z$$

with the same matrix A as in Section III and the matrix Q_3 is shown in (23). Following the same procedure as shown in the proof of Theorem 1, we conclude that *the system (6) with nonlinear particle interaction (3) is locally asymptotically stable at the equilibrium points* $(x_{i1}^*, 0)$ *if all of the eigenvalues of the matrix* Q_3 *defined in (23) have negative real parts; it is unstable if any of the eigenvalues of the matrix* Q_3 *has a positive real part.*

V. SIMULATION RESULTS

We have performed numerical simulations on arrays of different sizes ($3 \le N \le 256$). The system parameters used are $\gamma = 0.1, \kappa = 0.26$ ([18], [29]). Random initial conditions are used in the simulations.

First, we verify the stability of the open-loop frictional dynamics (6) in the presence of linear particle interactions. We use the system parameters as:

$$N = 3, \kappa = 0.26, \gamma = 0.1.$$

By solving (8), we have the following two sets of the equilibrium points for $(\phi_1, \dot{\phi}_1, \phi_2, \dot{\phi}_2, \phi_3, \dot{\phi}_3)$ at

$$\begin{aligned} Case1: &\quad (0.1941, 0, 0.9360, 0, 4.7747, 0) \\ Case2: &\quad (0.69, 0, 3.14, 0, 5.59, 0) \end{aligned}$$

For Case 1, we check that $\cos\phi_i > 0, i = 1, 2, 3$. From Theorem 1, we know that this set of equilibrium points is locally asymptotically stable. This claim is verified by Figure 2.

For Case 2, we check that $\cos\phi_i > 0, i = 1, 3$ but $\cos\phi_2 < 0$. We need to calculate the eigenvalues of Q_2 according to (15). We obtain:

$$Q_2 = \begin{bmatrix} -1.0312 & 0.2600 & 0 \\ 0.2600 & 0.4800 & 0.2600 \\ 0 & 0.2600 & -1.0292 \end{bmatrix} \quad (24)$$

whose eigenvalues are $-1.1150, -1.0302, 0.5648$. We can see that the second set of the equilibrium points is unstable. Figure 3 shows the unstability of the particle positions which converge to other points.

In the presence of nonlinear particle interactions, with the same system parameters κ, γ and $\beta = 1$, solving (21), we obtain the following two sets of the equilibrium points for $(\phi_1, \dot{\phi}_1, \phi_2, \dot{\phi}_2, \phi_3, \dot{\phi}_3)$ at:

$$\begin{aligned} Case1: &\quad (0.0001, 0, 0.0004, 0, 6.2827, 0) \\ Case2: &\quad (0.01, 0, 3.14, 0, 6.27, 0) \end{aligned}$$

For Case 1, we obtain Q_3 as defined in (23):

$$Q_3 = \begin{bmatrix} -1.2598 & 0.2598 & 0 \\ -0.0005 & -1.2593 & 0.2598 \\ 0 & 0.5195 & -1.5195 \end{bmatrix} \quad (25)$$

whose eigenvalues are all negative. Therefore, the first set of equilibrium points is asymptotically stable, which is shown in Figure 4.

For Case 2, we obtain Q_3 as the following:

$$Q_3 = \begin{bmatrix} -0.9896 & -0.0104 & 0 \\ -0.0104 & 1.0207 & -0.0104 \\ 0 & 0.5086 & -1.5085 \end{bmatrix} \quad (26)$$

whose eigenvalues have mixed signs. Therefore, the second set of equilibrium points is unstable. Figure 5 demonstrate it.

VI. CONCLUSION

We studied stability of coupled oscillators motivated by the FK model. The FK model characterizes the fundamental physics in many physical problems. The open-loop system of the FK model represents interconnected oscillators that have locally stable or unstable equilibrium points. We reveal the stability of the coupled system in the presence of linear and nonlinear particle interactions, respectively. Direct linearization is applied and a similarity transformation is found to facilitate the stability study of high-dimensional linear systems. The model can be applied to explain phenomena in atomic-scale friction and other systems whose dynamics can be described by the FK model.

$$Q_3 = \begin{bmatrix} -c_{11} - \cos x_{11}^* & c_{11} & 0 & \dots & 0 \\ c_{21} & -(c_{21} + c_{22} + \cos x_{21}^*) & c_{22} & 0 & \dots \\ & & \vdots & & \\ 0 & \dots & c_{N-1,1} & -(c_{N-1,1} + c_{N-1,2} + \cos x_{N-1,1}^*) & c_{N-1,2} \\ 0 & \dots & 0 & c_{N2} & -c_{N2} - \cos x_{N1}^* \end{bmatrix}. \tag{23}$$

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APPENDIX

Lemma for the Proof of Theorem 1

Lemma 1 ([30], page 171): If A is an $n \times n$ real symmetric matrix, then there always exist matrices L and D such that $L^T L = LL^T = I$ and $LAL^T = D$, where D is the diagonal matrix of eigenvalues of A.

Lemma 2 ([27]): Let $A = [a_{ij}]_{i,j=1}^n \in \Re^{n \times n}$ and assume that $a_{ii} > 0$ for each i and $a_{ij} \leq 0$ whenever $i \neq j$. If A is diagonally dominant, that is,

$$a_{ii} > \sum_{j=1, j \neq i}^{n} |a_{ij}|, \quad i = 1, 2, \dots, n,$$

or, if A is irreducible and

$$a_{ii} \geq \sum_{j=1, j \neq i}^{n} |a_{ij}|, \quad i = 1, 2, \dots, n,$$

with strict inequality for at least one i, then A is an M-matrix. A symmetric M-matrix is positive definite.

Lemma 3 ([31], Appendix A): Define the set W consisting of all zero row sum matrices which have only nonpositive off-diagonal elements. A matrix $A \in W$ satisfies:

1) All eigenvalues of A are nonnegative;
2) 0 is an eigenvalue of A;
3) 0 is an eigenvalue of multiplicity 1 if A is irreducible.

REFERENCES

[1] C. M. Breder. Equations descriptive of fish schools and other animal aggregations. *Ecology*, 35:361–370, 1954.

[2] C. W. Reynolds. Flocks, herds, and schools: A distributed behaviioral model. *Computer Graphics*, 21(4):25–34, 1987.

[3] D. Brogan and J. K. Hodgins. Group behaviors for systems with significant dynamics. *Autonomous Robots*, 4:137–153, 1997.

[4] A. Mogilner and L. Edelstein-Keshet. A non-local model for a swarm. *J. Math. Biol.*, 38:534–570, 1999.

[5] C. M. Topaz and A. L. Bertozzi. Swarming patterns in a two-dimensional kinematic model for biological groups. *SIAM J. Appl. Math.*, 65(1):152–174, 2004.

[6] V. Gazi and K. M. Passino. Stability analysis of swarms. *IEEE Trans. Automat. Contr.*, 48(4):692–696, 2003.

[7] N. Wiener. *Nonlinear Problems in Random Theory*. MIT Press, Cambridge, MA, 1958.

[8] A. T. Winfree. Biological rhythms and the behavior of populations of coupled oscillators. *Journal of Theoretical Biology*, 16(1):15–42, 1967.

[9] Y. Kuramoto. In H. Arakai, editor, *International Symposium on Mathematical Problems in Theoretical Physics*, Lecture Notes in Physics, page 420. Springer, New York, 1975.

[10] N. Shimoyama, K. Sugawara, T. Mizuguchi, Y. Hayakawa, and M. Sano. Collective motion in a system of motile elements. *Physical Review Letters*, 76(20), 1996.

[11] T. Vicsek, A. Czirok, E. Ben-Jacob, I. Cohen, and O. Shochet. Novel type of phase transition in a system of self-driven particles. *Physical Review Letters*, 75(6):1226–1229, 1995.

[12] S. H. Strogatz. From Kuramoto to Crawford: Exploring the onset of synchronization in populations of coupled oscillators. *Physica D*, 143:1–20, 2000.

[13] A. Jadbabaie, J. Lin, and A.S. Morse. Coordination of groups of mobile autonomous agents using nearest neighbor rules. *IEEE Trans. on Automatic Control*, 48(6):988–1001, 2003.

[14] R. Olfati-Saber and R. M. Murray. Consensus problems in networks of agents with switching topology and time-delays. *IEEE Trans. Automat. Contr.*, 49(9):1520–1533, 2004.

[15] Z. Qu, J. Wang, and R. A. Hull. Cooperative control of dynamical systems with application to autonomous vehicles. *IEEE Trans. Automat. Contr.*, to appear.

[16] E. Canale and P. Monzon. Gluing kuramoto coupled oscillators networks. In *Proceedings of the IEEE Conference on Decision and Control*, pages 4596–5601, New Orleans, LA, Dec. 2007.

[17] Y. I. Frenkel and T. Kontorova. *Zh. Eksp. Teor. Fiz.*, 8:1340, 1938.

[18] Y. Braiman, J. Barhen, and V. Protopopescu. Control of friction at the nanoscale. *Physical Review Letters*, 90(9), 2003.

[19] Y. Guo, Z. Qu, and Z. Zhang. Lyapunov stability and precise control of the frictional dynamics of a one-dimensional nanoarray. *Physical Review B*, 73(9):094118, 2006.

[20] G. Zhou and C. T. Zhang. A short review on the nonlinear motion in DNA. *Physica Scripta*, 43:347–352, 1991.

[21] O. M. Braun and Y. S. Kivshar. *The Frenkel-Kontorova Model*. Springer, Berlin Heidelberg, 2004.

[22] Y. Guo and Z. Qu. Control of frictional dynamics of a one-dimensional particle array. *Automatica*, 44:2560–2569, 2008.

[23] Y. Guo, Z. Qu, Y. Braiman, Z. Zhang, and J. Barhen. Nanotribology and nanoscale friction: Smooth sliding through feedback control. *IEEE Control Systems Magazine*, 28(6):92–100, 2008.

[24] C. I. Chou, C. L. Ho, B. Hu, and H. Lee. Morse-type frenkel-kontorova model. *Physical Review E*, 57(3):2747–2756, 1998.

[25] H. Khalil. *Nonlinear Systems*. Prentice Hall, 3rd edition, 2002.

[26] J. J. E. Slotine and W. Li. *Applied Nonlinear Control*. Prentics Hall, Nre Jersey, 1991.

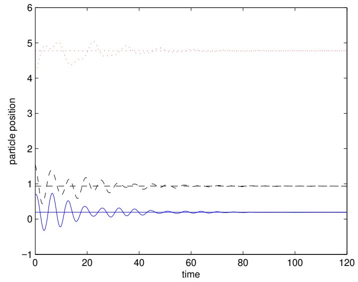
[27] P. Lancaster and M. Tismenetsky. *The Theory of Matrices with Applications*. Academic Press, second edition, 1985.

[28] Y. Guo and Z. Qu. Single particle dynamics and control in a sliding nanocluster system. In *Proceedings of the IEEE Conference on Decision and Control*, pages 2458–2463, New Orlean, Dec. 2007.

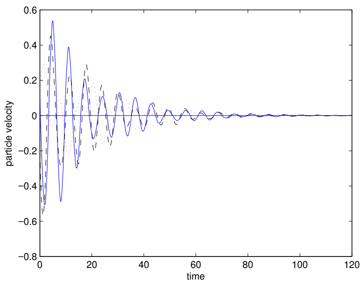
[29] Y. Guo and Z. Qu. Stabilization and tracking control of friction dynamics of a one-dimensional nanoarray. In *Proceedings of the American Control Conference*, pages 2487–2492, Portland, Oregon, June 2005.

[30] C. Godsil and G. Royle. *Algebraic Graph Theory*. Springer, New York, 2001.

[31] C. W. Wu. *Synchronization in Coupled Chaotic Circuits and Systems*. World Scientific, Singapore, 2002.

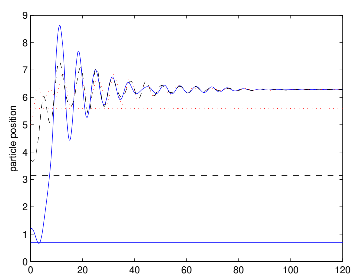


(a)

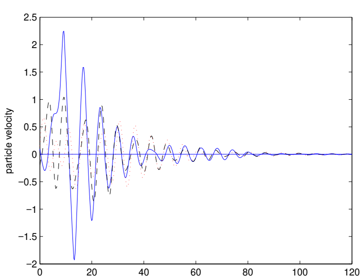


(b)

Fig. 2. Local stability of the equilibrium points $(\phi_1, \dot{\phi}_1, \phi_2, \dot{\phi}_2, \phi_3, \dot{\phi}_3) = (0.1941, 0, 0.9360, 0, 4.7747, 0)$ in the presence of linear particle interactions. (a) Particle positions; (b) Particle velocities.

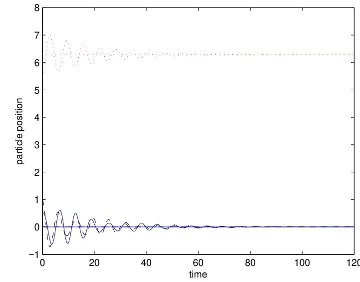


(a)

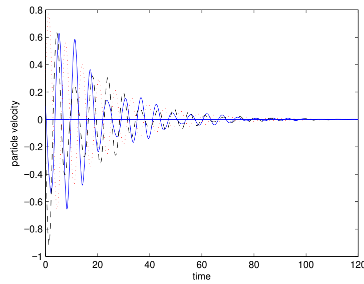


(b)

Fig. 3. Local unstability of the equilibrium points $(\phi_1, \dot{\phi}_1, \phi_2, \dot{\phi}_2, \phi_3, \dot{\phi}_3) = (0.69, 0, 3.14, 0, 5.59, 0)$ in the presence of linear particle interactions. (a) Particle positions; (b) Particle velocities.

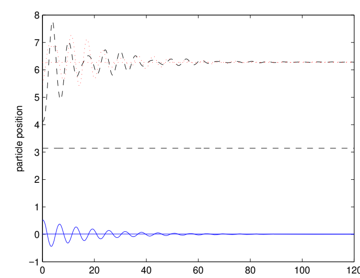


(a)

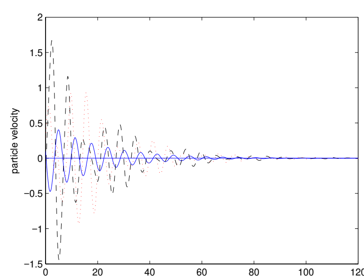


(b)

Fig. 4. Local stability of the equilibrium points $(\phi_1, \dot{\phi}_1, \phi_2, \dot{\phi}_2, \phi_3, \dot{\phi}_3) = (0.0001, 0, 0.0004, 0, 6.2827, 0)$ in the presence of nonlinear particle interactions. (a) Particle positions; (b) Particle velocities.



(a)



(b)

Fig. 5. Local unstability of the equilibrium points $(\phi_1, \dot{\phi}_1, \phi_2, \dot{\phi}_2, \phi_3, \dot{\phi}_3) = (0.01, 0, 3.14, 0, 6.27, 0)$ in the presence of nonlinear particle interactions. (a) Particle positions; (b) Particle velocities.