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Control of frictional dynamics of a one-dimensional particle array☆

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A B S T R A C T

Control of frictional forces is required in many applications of tribology. While the problem is approached by chemical means traditionally, a recent approach was proposed to control the system mechanically to tune frictional responses. We design feedback control laws for a one-dimensional particle array sliding on a surface subject to friction. The Frenkel–Kontorova model describing the dynamics is a nonlinear interconnected system and the accessible control elements are average quantities only. We prove local stability of equilibrium points of the un-controlled system in the presence of linear and nonlinear particle interactions, respectively. We then formulate a tracking control problem, whose control objective is for the average system to reach a designated targeted velocity using accessible elements. Sufficient stabilization conditions are explicitly derived for the closed-loop error systems using the Lyapunov theory based methods. Simulation results show satisfactory performances. The results can be applied to other physical systems whose dynamics is described by the Frenkel–Kontorova model.

1. Introduction

Tribology has been an active research area due to its broad applications in the fields of physics, chemistry, geology, biology, and engineering (Persson, 2000). Rapidly growing areas of tribology are in micro-electro-mechanical systems (MEMS), and biological systems, particularly the lubrication mechanisms in joints. Recent advances have substantially improved the understanding of frictional phenomena, particularly on the inherently nonlinear nature of friction (Urbakh, Klafter, Gourdon, & Israelachvili, 2004). Traditionally, the control of frictional forces has been approached by chemical means, such as supplementing base lubricants with friction modifier additives. A recent different approach, which tunes frictional responses by controlling the system mechanically via normal vibrations of small amplitude and energy, has attracted considerable interest, see Braiman, Barhen, and Protopopescu (2003), Cochard, Bureau, and Baumberger (2003), Gao, Luedtke, and Landman (1998), Heuberger, Drummond, and Israelachvili (1998), Rozman, Urbakh, and Klafter (1998) and Zaloj, Urbakh, and Klafter (1999). The idea is to reduce the frictional force or to eliminate stick-slip motion through a stabilization of desirable modes of motion. We follow this line of research and design feedback control laws to control frictional dynamics towards a desirable mode of motion.

Friction can be manipulated by applying perturbations to accessible elements and parameters of a sliding system (Persson, 2000). The authors in Braiman et al. (2003) proposed an intriguing idea to control the overall motion of an array of mechanically coupled objects sliding on a dissipative substrate via feedback control and tracking, and applied the idea to a particle array, with the frictional dynamics described by the Frenkel–Kontorova (FK) model. A control problem was formulated therein, and a global feedback control scheme was presented to render the system's output, the velocity of the center of mass of the nanoarray, to approach a given targeted value, subject to some fluctuations. Results were supported by simulations only. Theoretical justification on the non-Lipschitzian control was later given in Protopopescu and Barhen (2004). However, we showed in Guo, Qu, and Zhang (2006) that the control law in Braiman et al. (2003) does not eliminate the persistent oscillations of the controlled variables around their equilibrium points.

We study in this paper the problem of controlling frictional dynamics of a one-dimensional particle array using control theoretical methods. We describe the FK model to characterize the dynamics of the interconnected one-dimensional particle system. It is a nonlinear system since both the coupling of the particles with the substrate and the particle interactions are nonlinear. A control problem is then formulated based on the FK model, which is a constrained nonlinear control problem. The constraint is caused by the inaccessibility of individual particles. The control objective is to achieve tracking of the targeted velocity using physically accessible variables, *i.e.*, the average quantity of the interconnected system. We present two main results in the paper. First, we study stability of equilibrium points of the particle array in the presence of linear

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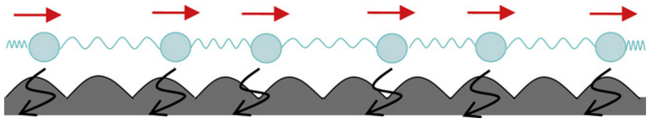

Fig. 1. The Frenkel–Kontorova model represents a harmonic chain (which mimics a layer of nano-particles) in a spatially periodic potential (which mimics the substrate). The chain is driven by a constant force which is damped by a velocity-proportional damping.

and nonlinear inter-particle coupling, respectively. Second, we design global tracking control laws to achieve that the average velocity of the array, *i.e.*, the velocity of the center mass, tracks any given constant targeted velocity. Global feedback control laws are explicitly constructed using the Lyapunov theory based method. We further analyze local stability of individual particles in the closed-loop system under the average control law. Finally, we illustrate the control performances using Matlab simulations of different sizes of a particle array. While the tracking control of the average system was presented in our early publication (Guo et al., 2006), further study on interconnected particle systems is shown in this paper, and sufficient conditions will be given to stabilize individual particle systems around the targeted trajectory under the average control. Also, we provide rigorous proof of stability at the equilibrium points of the open-loop interconnected system for the first time in this paper.

The paper is organized as follows. Section 2 presents the Frenkel–Kontorova model used to describe the frictional dynamics. In Section 3, the local stability of the open-loop interconnected particle system is analyzed in two subsections with linear and nonlinear particle interactions, respectively. Then, a tracking control problem is defined in Section 4. Section 5 presents tracking control design to solve the control problem formulated in Section 4. Simulation results are given in Section 6. The paper is finally concluded with brief remarks in Section 7.

Notations: $\|x\|$ denotes the Euclidean norm of vector x. x^{T} denotes the transpose of vector x. I_N denotes the identity matrix of dimension N. $\mathrm{diag}\{A_i\}$ denotes a block diagonal matrix whose diagonal elements are A_i. $\otimes$ denotes the Kronecker product:

$$A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B & \dots & a_{1p}B \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1}B & a_{n2}B & \dots & a_{np}B \end{bmatrix}$$

where A is an $n \times p$ matrix and B is an $m \times q$ matrix. Some useful properties of Kronecker product are given in the Appendix.

2. The Frenkel–Kontorova model

The basic equations for the driven dynamics of a one-dimensional particle array of N identical particles moving on a surface are given by a set of coupled nonlinear equations (Braiman et al., 2003; Braiman, Family, & Hentschel, 1997):

$$m'\ddot{z}_i + \gamma'\dot{z}_i = -\frac{\partial U(z_i)}{\partial z_i} - \sum_{\substack{j=1\\ j\neq i}}^{N} \frac{\partial W(z_i - z_j)}{\partial z_i} + f_i' + \eta(t), \tag{1}$$

where $i = 1, \dots, N$, z_i is the coordinate of the ith particle, m' is its mass, γ' is the positive friction coefficient representing the single particle energy exchange with the substrate, f_i' is the applied external force, $\eta(t)$ denotes additive the Gaussian noise, $U(z_i)$ is the periodic potential applied by the substrate, and $W(z_i - z_j)$ is the inter-particle interaction potential.

Under the simplifications that the substrate potential is in the form of $\frac{m'}{4\pi^2}(1 - \cos\frac{2\pi z_i}{a})$ with $a > 0$, the same force is applied to each particle, and there is zero noise (*i.e.*, $\eta(t) = 0$), the equation of motion reduces to the following FK model:

$$\ddot{\phi}_i + \gamma\dot{\phi}_i + \sin(\phi_i) = f + F_i \tag{2}$$

where ϕ_i is the dimensionless phase variable, $\phi_i = 2\pi z_i/a$, $\gamma = \gamma'/m'$, $f = 2\pi a f_i'/m'$,

$$F_i = -\frac{2\pi a}{m'} \sum_{j=1, j\neq i}^{N} \frac{\partial W(a\phi_i/2\pi - a\phi_j/2\pi)}{\partial(a\phi_i/2\pi)}.$$

A specific example often considered for the particle interaction force, F_i, is the nearest-neighbor interaction in the form of Morse-type interaction (Braiman et al., 2003, 1997):

$$F_i = \frac{\kappa}{\beta}\left\{e^{-\beta(\phi_{i+1}-\phi_i)} - e^{-2\beta(\phi_{i+1}-\phi_i)}\right\} - \frac{\kappa}{\beta}\left\{e^{-\beta(\phi_i-\phi_{i-1})} - e^{-2\beta(\phi_i-\phi_{i-1})}\right\}, \quad i = 2, \dots, N-1, \tag{3}$$

where κ and β are positive constants. The free-end boundary conditions are represented as:

$$F_1 = \frac{\kappa}{\beta}\left\{e^{-\beta(\phi_2-\phi_1)} - e^{-2\beta(\phi_2-\phi_1)}\right\},$$

$$F_N = -\frac{\kappa}{\beta}\left\{e^{-\beta(\phi_N-\phi_{N-1})} - e^{-2\beta(\phi_N-\phi_{N-1})}\right\}. \tag{4}$$

As $\beta \to 0$, (3) turns to:

$$F_i = \kappa\,(\phi_{i+1} - 2\phi_i + \phi_{i-1}), \quad i = 2, \dots, N-1, \tag{5}$$

which represents a linear approximation of particle interaction for small β with the following free-end boundary conditions:

$$F_1 = \kappa(\phi_2 - \phi_1), \qquad F_N = \kappa(\phi_{N-1} - \phi_N). \tag{6}$$

An illustration of the Frenkel–Kontorova model is shown in Fig. 1.

The FK model (2) describes a chain of particles interacting with the nearest neighbors in the presence of an external periodic potential. It is one of the best known simple models for frictional dynamics, and can be extended to two-dimensional and three-dimensional models and to a full set of molecular dynamics. Besides describing the frictional dynamics (Persson, 2000), the FK model has been widely involved in descriptions of many other physical problems, such as charge-density waves, magnetic spirals, and absorbed monolayers (Braun & Kivshar, 2004).

The FK model presents a nonlinear interconnected systems. Nonlinearity appears since (i) the coupling of the particles with the substrate is nonlinear, and (ii) the particle interaction is nonlinear. Particularly, the nonlinear Morse-type interaction represents an attraction force between two nearest particles when their distance is longer than the natural length of the spring, and a restoring force (increasing unlimited) between them when the distance is shorter than the natural length of the spring (Chou, Ho, Hu, & Lee, 1998). The Morse-type particle interaction presents a class of attraction/repulsive functions for a one-dimensional swarm aggregation, which may be of interest to the research in swarm dynamics, see Gazi and Passino (2002, 2003).

3. Open-loop stability analysis

Before we define our control problem, we study the stability of the open-loop system of the FK model (2). The dynamics in (2) expressed without external forces can be equivalently written as:

$$\begin{aligned}\dot{x}_{i1} &= x_{i2}\\ \dot{x}_{i2} &= -\sin x_{i1} - \gamma x_{i2} + F_i\end{aligned} \tag{7}$$

where $i = 1, 2 \dots, N$, $x_{i1} = \phi_i$, $x_{i2} = \dot{\phi}_i$, and F_i is the Morse-type particle interaction. Let us look at the local stability of the equilibrium points in the presence of (i) linear particle interaction, (ii) nonlinear particle interaction, respectively.

3.1. Linear particle interactions

We consider the local stability of (7) when F_i takes the form of the linear interaction given in (5). From (7), the equilibrium points are at $(x_{i1}, x_{i2}) = (x_{i1}^*, 0)$ where x_{i1}^* are solutions to

$$Q = \begin{bmatrix} -\kappa - \cos x_{11}^* & \kappa & 0 & \dots & 0 \\ \kappa & -2\kappa - \cos x_{21}^* & \kappa & 0 & \dots \\ & & \vdots & & \\ 0 & \dots & \kappa & -2\kappa - \cos x_{N-1,1}^* & \kappa \\ 0 & \dots & 0 & \kappa & -\kappa - \cos x_{N1}^* \end{bmatrix} \in \Re^{N\times N}.$$

Box I.

$$\begin{aligned} &-\sin x_{11}^* + \kappa(x_{21}^* - x_{11}^*) = 0, \\ &-\sin x_{i1}^* + \kappa(x_{i+1,1}^* - 2x_{i1}^* + x_{i-1,1}^*) = 0, \quad i = 2, \dots, N-1, \\ &-\sin x_{N1}^* + \kappa(x_{N-1,1}^* - x_{N1}^*) = 0. \end{aligned} \tag{8}$$

Define new states as $z_{i1} = x_{i1} - x_{i1}^*$, $z_{i2} = x_{i2}$, and linearize it around its equilibrium. We obtain

$$\begin{aligned} \dot{z}_{i1} &= z_{i2} \\ \dot{z}_{i2} &= -\cos x_{i1}^* z_{i1} - \gamma z_{i2} + \kappa(z_{i+1,1} - 2z_{i1} + z_{i-1,1}) \\ &\quad - \sin x_{i1}^* + \kappa(x_{i+1,1}^* - 2x_{i1}^* + x_{i-1,1}^*) \\ &= -\cos x_{i1}^* z_{i1} - \gamma z_{i2} + \kappa(z_{i+1,1} - 2z_{i1} + z_{i-1,1}). \end{aligned} \tag{9}$$

Note that the last equal sign holds due to the equilibrium equation (8).

Stacking the state space equations for $i = 1, 2, \dots, N$, we obtain

$$\dot{z} = \overline{A}z + \overline{B}\,\overline{Q}z \tag{10}$$

where $z = [z_{11},\ z_{12},\ z_{21}\ z_{22},\ \dots\ z_{N1},\ z_{N2}]^{\mathrm{T}}$,

$$\overline{A} = I_N \otimes A, \qquad \overline{B} = I_N \otimes B, \qquad \overline{Q} = Q \otimes \begin{bmatrix} 1 & 0 \end{bmatrix}, \tag{11}$$

and

$$A = \begin{bmatrix} 0 & 1 \\ 0 & -\gamma \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \tag{12}$$

and Q is represented in Box I. Note that $\overline{A}$ and $\overline{Q}$ can be represented by other matrices. We represented them as the current form for the convenience of the proof of Theorem 1 presented in the following.

Theorem 1. *The system (7) with linear particle interaction (5) is locally asymptotically stable at the equilibrium points $(x_{i1}^*, 0)$ if all of the eigenvalues of the matrix Q defined in Box I have negative real parts; it is unstable if any of the eigenvalues of the matrix Q has a positive real part. Particularly, it is locally asymptotically stable if $\cos x_{i1}^* \geq 0$ for all i with strict inequality for at least one i, and it is unstable if $\cos x_{i1}^* \leq 0$ for all i with strict inequality for at least one i.*

Next, we present the proof of Theorem 1, which needs the following three Lemma.

Lemma 1 (Godsil and Royle (2001, page 171) Spectral Theorem for Symmetric Matrices). *If A is an $n \times n$ real symmetric matrix, then there always exist matrices L and D such that $L^{\mathrm{T}}L = LL^{\mathrm{T}} = I$ and $LAL^{\mathrm{T}} = D$, where D is the diagonal matrix of eigenvalues of A.*

Lemma 2 (Lancaster & Tismenetsky, 1985). *Let $A = [a_{ij}]_{i,j=1}^n \in \Re^{n\times n}$ and assume that $a_{ii} > 0$ for each i and $a_{ij} \leq 0$ whenever $i \neq j$. If A is diagonally dominant, that is,*

$$a_{ii} > \sum_{j=1, j\neq i}^{n} |a_{ij}|, \quad i = 1, 2, \dots, n,$$

or, if A is irreducible and

$$a_{ii} \geq \sum_{j=1, j\neq i}^{n} |a_{ij}|, \quad i = 1, 2, \dots, n,$$

with strict inequality for at least one i, then A is an M-matrix. A symmetric M-matrix is positive definite.

Lemma 3 (Wu, 2002, Appendix A). *Define the set W consisting of all zero row sum matrices which have only nonpositive off-diagonal elements. A matrix $A \in W$ satisfies:*

(1) *All eigenvalues of A are nonnegative;*
(2) *0 is an eigenvalue of A;*
(3) *0 is an eigenvalue of multiplicity 1 if A is irreducible.*

Proof of Theorem 1. First, we study stability of the linearized system (10) for any positive constants γ, κ and for any $N \geq 2$. To perform this stability analysis, we find a transformation matrix to transform the system matrix into a block diagonal one.

Define a similarity transformation $z = \overline{T}\zeta$. In the new coordinate, the system dynamics is

$$\dot{\zeta} = H\zeta. \tag{13}$$

We show how to choose $\overline{T}$, and present H accordingly.

Since Q is a real symmetric matrix, according to Lemma 1, there exists a unitary matrix T such that $T^{-1}QT = D$ where D is a diagonal matrix of eigenvalues of Q. Let

$$\overline{T} = T \otimes I_2 \tag{14}$$

where I_2 is the 2×2 identity matrix. Then:

$$\begin{aligned} H &= \overline{T}^{-1}(\overline{A} + \overline{B}\,\overline{Q})\overline{T} \\ &= \overline{T}^{-1}\left[I_N \otimes A + (I_N \otimes B)\left(Q \otimes \begin{bmatrix} 1 & 0 \end{bmatrix}\right)\right]\overline{T} \\ &= \overline{T}^{-1}\left(I_N \otimes A + Q \otimes \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right)\overline{T} \\ &= \left(T^{-1}I_N T\right) \otimes A + \left(T^{-1}QT\right) \otimes \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \\ &= I_N \otimes A + D \otimes \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}. \end{aligned} \tag{15}$$

We can see that H is block diagonal, and the block diagonal element of H writes:

$$H_{ii} = \begin{bmatrix} 0 & 1 \\ \alpha_i & -\gamma \end{bmatrix}, \tag{16}$$

where α_i, $i = 1, 2, \dots, N$ are eigenvalues of Q. The stability of the system depends on the sign of the real parts of α_i, $i = 1, \dots, N$:

(1) If α_i, $i = 1, 2, \dots, N$ have negative real parts, the eigenvalues of H_{ii}, $i = 1, 2, \dots, N$ have also negative real parts, and so does the matrix H. This indicates that the system is asymptotically stable at these points. Due to the similarity transformation, the same stability result holds for the original system $\dot{z} = (\overline{A} + \overline{B}\,\overline{Q})z$. Furthermore, local stability of the original nonlinear system (7) can be deduced from the stability analysis of its linearized system (10) (Slotine & Li, 1991, Theorem 3.1).
(2) If α_i has a positive real part for any $i \in [1, N]$, eigenvalues of H_{ii}, $i = 1, 2, \dots, N$, also have positive real parts. With the same arguments as above, the system (7) is unstable at these points.

Checking the structure of matrix Q in Box I, we have the following cases:

- If $\cos x_{i1}^* \geq 0$ for all i with strict inequality for at least one i, the matrix $-Q$ is an M-matrix and $\alpha_i < 0$ for all i

$$Q=\begin{bmatrix}\kappa-\cos x_{11}^* & \kappa & 0 & \dots & 0\\ \kappa & -\cos x_{21}^* & \kappa & 0 & \dots\\ & & \vdots & & \\ 0 & \dots & \kappa & -\cos x_{N-1,1}^* & \kappa\\ 0 & \dots & 0 & \kappa & \kappa-\cos x_{N1}^*\end{bmatrix}+(-2\kappa)I_N \stackrel{\text{def}}{=} \Phi+(-2\kappa)I_N.$$

Box II.

$$Q=\begin{bmatrix}-c_{11}-\cos x_{11}^* & c_{11} & 0 & \dots & 0\\ c_{21} & -(c_{21}+c_{22}+\cos x_{21}^*) & c_{22} & 0 & \dots\\ & & \vdots & & \\ 0 & \dots & c_{N-1,1} & -(c_{N-1,1}+c_{N-1,2}+\cos x_{N-1,1}^*) & c_{N-1,2}\\ 0 & \dots & 0 & c_{N2} & -c_{N2}-\cos x_{N1}^*\end{bmatrix}.$$

Box III.

according to Lemma 2. Therefore, Q is Hurwitz and the system is asymptotically stable;

- If $\cos x_{i1}^*=0$ for all i, Q has one (and only one) eigenvalue 0 according to Lemma 3. The linear system (10) is marginally stable and the stability of the nonlinear system (7) could be either stable or unstable;
- If $\cos x_{i1}^*\leq 0$ for all i with strict inequality for at least one i, we can represent Q as in Box II. Since Φ is an irreducible and nonnegative matrix, it has a positive eigenvalue, r, equal to the spectral radius of Φ, which is between $2k+\min\{-\cos x_{11}^*,\dots,-\cos x_{N1}^*\}$ and $2k+\max\{-\cos x_{11}^*,\dots,-\cos x_{N1}^*\}$ (Lancaster & Tismenetsky, 1985, page 537). Therefore, Q has at least one positive eigenvalue. The system is unstable;
- If $\cos x_{i1}^*, i=1,\dots,N$, have mixed signs, the system could be either stable or unstable and numerical calculations is necessary to determine the sign of the real parts of the eigenvalues of Q. □

Remark 1. As special cases of Theorem 1, the equilibrium points $(2k\pi,0), k=0,\pm1,\dots$, are asymptotically stable and $((2k+1)\pi,0)$ are unstable. The result was first claimed in our early publication (Guo et al., 2006) without rigorous proof. We extend the result to include all equilibrium points of the open-loop system in this paper.

3.2. Nonlinear particle interactions

In the presence of Morse-type nonlinear particle interactions, that is, F_i takes the form (3), the equilibrium points of (7) are at $(x_{i1},x_{i2})=(x_{i1}^*,0)$ where x_{i1}^* are solutions to

$$\begin{aligned}&-\sin x_{11}^*+\frac{\kappa}{\beta}\left\{e^{-\beta(x_{21}^*-x_{11}^*)}-e^{-2\beta(x_{21}^*-x_{11}^*)}\right\}=0,\\ &-\sin x_{i1}^*+\frac{\kappa}{\beta}\left\{e^{-\beta(x_{i+1,1}^*-x_{i1}^*)}-e^{-2\beta(x_{i+1,1}^*-x_{i1}^*)}\right\}\\ &\quad-\frac{\kappa}{\beta}\left\{e^{-\beta(x_{i1}^*-x_{i-1,1}^*)}-e^{-2\beta(x_{i1}^*-x_{i-1,1*})}\right\}=0,\\ &\quad i=2,\dots,N-1,\\ &-\sin x_{N1}^*-\frac{\kappa}{\beta}\left\{e^{-\beta(x_{N1}^*-x_{N-1,1}^*)}-e^{-2\beta(x_{N1}^*-x_{N-1,1}^*)}\right\}=0.\end{aligned}\tag{17}$$

Let $z_{i1}=x_{i1}-x_{i1}^*, z_{i2}=x_{i2}$, and linearize the system around its equilibrium. After simplification, we get

$$\begin{aligned}\dot z_{i1}&=z_{i2}\\ \dot z_{i2}&=-\cos x_{i1}^*z_{i1}-\gamma z_{i2}\\ &\quad+\frac{\kappa}{\beta}\left[-e^{-\beta(x_{i+1,1}^*-x_{i1}^*)}+2e^{-2\beta(x_{i+1,1}^*-x_{i1}^*)}\right](z_{i+1,1}-z_{i1})\\ &\quad-\frac{\kappa}{\beta}\left[-e^{-\beta(x_{i,1}^*-x_{i-1,1}^*)}+2e^{-2\beta(x_{i1}^*-x_{i-1,1}^*)}\right](z_{i1}-z_{i-1,1})\\ &\stackrel{\text{def}}{=}-\cos x_{i1}^*z_{i1}-\gamma z_{i2}+c_{i1}(z_{i+1,1}-z_{i1})-c_{i2}(z_{i1}-z_{i-1,1}).\end{aligned}\tag{18}$$

We can see that (18) is in the same form as in (9) with different coupling coefficients. We can represent (18) as

$$\dot z=\overline{A}z+\overline{B}\,\overline{Q}z$$

with the same forms of the matrices $\overline{A},\overline{B},\overline{Q}$ as in Section 3.1 but different matrix Q shown in Box III.

Following the same procedure as shown in the proof of Theorem 1, we conclude that *the system (7) with nonlinear particle interaction (3) is locally asymptotically stable at the equilibrium points $(x_{i1}^*,0)$ if all of the eigenvalues of the matrix Q defined in Box III have negative real parts; it is unstable if any of the eigenvalues of the matrix Q has a positive real part.*

In the next section, we formulate our control problem and then discuss the control design in the subsequent section.

4. Control problem formulation

Control can be applied to the particle array, so that the frictional dynamics of a small array of particles is controlled towards preassigned values of the average sliding velocity. Let the external force, f, in (2) be a feedback control, denoted by $u(t)$. Rewrite the system model (2) as follows (Braiman et al., 2003):

$$\ddot\phi_i+\gamma\dot\phi_i+\sin(\phi_i)=F_i+u(t).\tag{19}$$

Due to physical accessibility constraints, the feedback control $u(t)$ is a function of three measurable quantities, v_{target}, $v_{c.m.}$, and $\phi_{c.m.}$, where v_{target} is the constant targeted velocity for the center of mass, $v_{c.m.}$ is the average (center of mass) velocity, *i.e.*,

$$v_{c.m.}=\frac{1}{N}\sum_{i=1}^N\dot\phi_i,\tag{20}$$

and $\phi_{c.m.}$ is the average (center of mass) position, *i.e.*,

$$\phi_{c.m.}=\frac{1}{N}\sum_{i=1}^N\phi_i.\tag{21}$$

We define the following tracking control problem:

Design a feasible feedback control law

$$u(t)=u(v_{\text{target}},v_{c.m.},\phi_{c.m.}),\tag{22}$$

such that $v_{c.m.}$ tends to v_{target}.

In nanoscale friction control, it is sufficient to control the system as a whole. It can be seen that the tracking control problem is a constrained control problem since the accessible variables are average quantities only. Existing results in nonlinear decentralized control (for example, Guo, Jiang, and Hill (1999), Ioannou (1986), Jiang (2002) and Tezcan and Basar (1999)) cannot be applied due to the inaccessibility of the subsystems' states.

In the next section, we first construct feedback control laws to solve the tracking control problem defined above, and we further analyze the single particle stability in the closed-loop system under the designed average control law.

5. Tracking control design

To design feedback tracking controllers, we define the following tracking error states:

$$e_{i1} = \phi_i - v_{\text{target}} t, \qquad e_{i2} = \dot{\phi}_i - v_{\text{target}}. \tag{23}$$

The corresponding error dynamics for a single particle is given as:

$$\dot{e}_{i1} = e_{i2}$$
$$\dot{e}_{i2} = -\sin(e_{i1} + v_{\text{target}} t) - \gamma(e_{i2} + v_{\text{target}}) + F_i + u(t). \tag{24}$$

5.1. Tracking control of the average system

In this subsection, we design tracking control to solve the problem defined in Section 4, which is to render the average velocity of the system, *i.e.* the velocity of the center of the mass, to converge to a constant targeted value. To this end, we introduce the average error states as:

$$e_{1av} = \phi_{c.m.} - v_{\text{target}} t, \qquad e_{2av} = v_{c.m.} - v_{\text{target}}, \tag{25}$$

where $v_{c.m.}$ and $\phi_{c.m.}$ are defined in (20) and (21), respectively. Then, it is obvious that the convergence of $(\phi_{c.m.}, v_{c.m.})$ to $(v_{\text{target}} t, v_{\text{target}})$ is equivalent to the convergence of (e_{1av}, e_{2av}) to $(0, 0)$. Therefore, asymptotic stability of the system in the error state space is equivalent to asymptotic tracking of the targeted positions and constant velocity.

The dynamics of (e_{1av}, e_{2av}) can be derived as:

$$\dot{e}_{1av} = e_{2av}$$
$$\dot{e}_{2av} = -\frac{1}{N}\sum_{i=1}^{N} \sin(e_{i1} + v_{\text{target}} t) - \gamma(e_{2av} + v_{\text{target}}) + u(t). \tag{26}$$

Note that the F_i term disappeared in (26) because the sum of F_i is zero for Morse-type interactions of the form defined in (3).

We construct the following Lyapunov function candidate:

$$W(e_{av}) = \frac{1}{2}e_{1av}^2 + \frac{1}{2}(c_1 e_{1av} + e_{2av})^2 \tag{27}$$

where c_1 is a positive design constant, and $e_{av} = [e_{1av} \;\; e_{2av}]^{\mathrm{T}}$.

Taking the time derivative of W along the dynamics of (26), and denoting

$$\xi = c_1 e_{1av} + e_{2av}, \tag{28}$$

we have:

$$\dot{W}(e_{av}) = -c_1 e_{1av}^2 + \xi \Bigg[e_{1av} + c_1 e_{2av} - \gamma e_{2av} - \frac{1}{N}\sum_{i=1}^{N} \sin(e_{i1} + v_{\text{target}} t) - \gamma v_{\text{target}} + u(t) \Bigg]. \tag{29}$$

Choose

$$\begin{aligned} u(t) &= \gamma v_{\text{target}} - e_{1av} - (c_1 - \gamma) e_{2av} \\ &\quad - (c_1 + c_2)\xi + \sin(v_{\text{target}} t) \\ &= \gamma v_{\text{target}} - k_1(\phi_{c.m.} - v_{\text{target}} t) \\ &\quad - k_2(v_{c.m.} - v_{\text{target}}) + \sin(v_{\text{target}} t) \end{aligned} \tag{30}$$

where c_2 is a positive design constant, $k_1 = 1 + (c_1 + c_2)c_1$, $k_2 = 2c_1 + c_2 - \gamma$, and the term $\sin(v_{\text{target}} t)$ is introduced to enforce the equilibrium of the closed-loop system (26) to be the origin.

We obtain:

$$\begin{aligned} \dot{W}(e_{av}) &= -c_1(e_{1av}^2 + \xi^2) - c_2\xi^2 \\ &\quad + \xi \frac{1}{N}\sum_{i=1}^{N} \left[-\sin(e_{i1} + v_{\text{target}} t) + \sin(v_{\text{target}} t) \right] \\ &\le -c_1(e_{1av}^2 + \xi^2) - c_2\xi^2 + |\xi| \\ &\quad \times \frac{1}{N}\sum_{i=1}^{N} \left| -\sin(e_{i1} + v_{\text{target}} t) + \sin(v_{\text{target}} t) \right| \\ &\le -c_1(e_{1av}^2 + \xi^2) - c_2\xi^2 + 2|\xi|. \end{aligned} \tag{31}$$

Since the maximum of the last two terms is $1/c_2$, we have

$$\dot{W}(e_{av}) \le -c_1(e_{1av}^2 + \xi^2) + \frac{1}{c_2}, \tag{32}$$

which can be used to prove uniform boundedness of the error system (26) as shown in the proof of Theorem 2.

To achieve asymptotical tracking, that is, to make the error system (26) asymptotically stable, the following switching control law can be used:

$$\begin{aligned} u(t) &= \gamma v_{\text{target}} - k_1(\phi_{c.m.} - v_{\text{target}} t) - k_2\left(v_{c.m.} - v_{\text{target}}\right) \\ &\quad + \sin(v_{\text{target}} t) - 2\,\mathrm{sgn}(\xi) \end{aligned} \tag{33}$$

where $\mathrm{sgn}(\xi)$ denotes the signum function, defined as $\mathrm{sgn}(\xi) = 1$ for $\xi > 0$, $\mathrm{sgn}(\xi) = -1$ for $\xi < 0$, and $\mathrm{sgn}(\xi) = 0$ for $\xi = 0$.

The following theorem presents the stability results of the closed-loop average error system (26).

Theorem 2. *The feedback control laws (30) or (33) solve the tracking control of the average system defined in Section 4. Using (30), the tracking error between the velocity of the center of mass and the targeted velocity is uniformly bounded over time $[0, \infty)$. Under the switching control law (33), the tracking error goes to zero asymptotically.*

Proof. Using the continuous control law (30), for the positive definite Lyapunov function W defined in (27), we obtained (32). Then,

$$\dot{W}(e_{av}) \le 0, \quad \forall \|(e_{1av}, \xi)\| \ge \frac{1}{\sqrt{c_1 c_2}}. \tag{34}$$

We conclude that the solutions of the closed-loop systems (26), (30) are globally uniformly bounded.

To calculate the ultimate bound, we notice from (27) that

$$\begin{aligned} \frac{1}{2}\lambda_{\min}(P)\|e_{av}\|^2 &\le W(e_{av}) = \frac{1}{2}e_{av}^{\mathrm{T}} P e_{av} \\ &\le \frac{1}{2}\lambda_{\max}(P)\|e_{av}\|^2 \end{aligned} \tag{35}$$

where $e_{av} = [e_{1av} \; e_{2av}]^{\mathrm{T}}$,

$$P = \begin{bmatrix} 1 + c_1^2 & c_1 \\ c_1 & 1 \end{bmatrix},$$

and $\lambda_{\min}(P)$, and $\lambda_{\max}(P)$ denote the minimum and maximum eigenvalues of the matrix P, respectively. From (35), we have

$$\|e_{av}\|^2 \le \frac{2W(e_{av})}{\lambda_{\min}(P)} = \frac{\|(e_{1av}, \xi)\|^2}{\lambda_{\min}(P)}. \tag{36}$$

Due to (34), we obtain

$$\dot{W}(e_{av}) \le 0, \quad \forall \|e_{av}\| \ge \frac{1}{\sqrt{c_1 c_2 \lambda_{\min}(P)}}. \tag{37}$$

The ultimate bound of $\|e_{av}\|$ is given by Khalil (2002, Section 4.8):

$$b = \sqrt{\frac{\lambda_{\max}(P)}{c_1 c_2 \lambda_{\min}^2(P)}}. \tag{38}$$

By choosing c_1, c_2 appropriately (with the price of a large control effort), we can have the error states to be arbitrarily close to zero.

Under the switching control law (33) (which is the continuous control (30) plus a switching term), substituting (33) into (29), we get

$$\begin{aligned}\dot{W}(e_{av}) &\leq -c_1(e_{1av}^2+\xi^2)-c_2\xi^2+2|\xi|-\xi 2\mathrm{sgn}(\xi)\\ &\leq -c_1(e_{1av}^2+\xi^2),\end{aligned} \quad (39)$$

which is negative definite. Asymptotic stability of the error system follows from Lyapunov theory. □

The two control laws (30) and (33) were first presented in our early publication (Guo et al., 2006) without rigorous proof.

It should be noted that the controller proposed in (30) render the velocity of the average system to go to the targeted value while the individual particles could have different modes of motion. Next, we investigate stability of single particles in the closed-loop system under the average control law (30).

5.2. Stability of single particles in the closed-loop system

We assume linear particle interactions in this subsection. From (5) and (23), representing F_i using the error states, we have:

$$\begin{aligned}&F_i=\kappa\left(e_{i+1,1}-2e_{i1}+e_{i-1,1}\right),\quad i=2,\ldots,N-1,\\ &F_1=\kappa(e_{21}-e_{11}), F_N=\kappa(e_{N-1,1}-e_{N1}).\end{aligned} \quad (40)$$

For the convenience of presentation, let

$$\bar{k}_1=\frac{k_1}{N},\qquad \bar{k}_2=\frac{k_2}{N}. \quad (41)$$

Substituting the control law defined in (30) into (24), we have the state space model of the closed-loop system in the following form:

$$\begin{aligned}\dot{e}_{i1}&=e_{i2}\\ \dot{e}_{i2}&=-\gamma e_{i2}+F_i-\bar{k}_1\left(\sum_{j=1}^N e_{j1}\right)-\bar{k}_2\left(\sum_{j=1}^N e_{j2}\right)\\ &\quad+\left[\sin(\upsilon_{\text{target}}t)-\sin(e_{i1}+\upsilon_{\text{target}}t)\right].\end{aligned} \quad (42)$$

Linearize the system around the equilibrium $e^*=0$, where $e=[e_{11},\ e_{12},\ e_{21},\ e_{22},\ \ldots,\ e_{N1},\ e_{N2}]^{\mathrm{T}}$. Since

$$\begin{aligned}&\sin(\upsilon_{\text{target}}t)-\sin(e_{i1}+\upsilon_{\text{target}}t)\\ &\quad=-2\sin\frac{e_{i1}}{2}\cos\frac{e_{i1}+2\upsilon_{\text{target}}t}{2},\end{aligned} \quad (43)$$

we obtain the following linearized model:

$$\dot{e}=Ge, \quad (44)$$

where

$$\begin{aligned}G&=I_N\otimes\begin{bmatrix}0&1\\0&-\gamma\end{bmatrix}+Q\otimes\begin{bmatrix}0&0\\\kappa&0\end{bmatrix}+\Theta\otimes\begin{bmatrix}0&0\\-\bar{k}_1&-\bar{k}_2\end{bmatrix},\\ &\quad+I_N\otimes\begin{bmatrix}0&0\\-\cos\upsilon_{\text{target}}t&0\end{bmatrix}\\ &=I_N\otimes\begin{bmatrix}0&1\\-\cos\upsilon_{\text{target}}t&-\gamma\end{bmatrix}+Q\otimes\begin{bmatrix}0&0\\\kappa&0\end{bmatrix}\\ &\quad+\Theta\otimes\begin{bmatrix}0&0\\-\bar{k}_1&-\bar{k}_2\end{bmatrix}\end{aligned} \quad (45)$$

where Θ is the N by N matrix of ones, and

$$Q=\begin{bmatrix}-1&1&0&\ldots&0\\1&-2&1&0&\ldots\\&&\vdots&&\\0&\ldots&1&-2&1\\0&\ldots&0&1&-1\end{bmatrix}. \quad (46)$$

We have the following lemma:

Lemma 4. *There exists a similarity transformation such that the matrix G in (44) can be transformed to a block diagonal one.*

Proof of Lemma 4. Notice that the matrix $(-Q)$ is a real symmetric matrix with zero row sum, and it is irreducible. From Lemmas 1 and 3, $(-Q)$ has eigenvalues

$$\mu_1\geq\mu_2\geq\cdots\geq\mu_{N-1}>\mu_N=0. \quad (47)$$

It is always possible to choose the eigenvectors to be real, normalized and mutually orthogonal. Denote the eigenvectors corresponding to each of the eigenvalues:

$$\upsilon_k=[\upsilon_{1k},\ \upsilon_{2k},\ \ldots,\ \upsilon_{Nk}],\quad k=1,2,\ldots,N-1;\ \upsilon_N. \quad (48)$$

Then $V=[\upsilon_1\ \upsilon_2\ \ldots\ \upsilon_N]$ is an orthogonal matrix, *i.e.*, $VV^{\mathrm{T}}=V^{\mathrm{T}}V=I$, implying $V^{\mathrm{T}}=V^{-1}$, and

$$\sum_{k=1}^N\upsilon_{ki}\upsilon_{kj}=\sum_{k=1}^N\upsilon_{ik}\upsilon_{jk}=\delta_{ij}, \quad (49)$$

where $\delta_{ij}=1$ for $i=j$ and $\delta_{ij}=0$ for $i\neq j$. Because of $-V^{\mathrm{T}}QV=\mathrm{diag}(\mu_1,\mu_2,\ldots,\mu_N)$, we further have

$$(-Q)_{ij}=\sum_{k=1}^N\mu_k\upsilon_{ik}\upsilon_{jk}. \quad (50)$$

Because the eigenvectors υ_k, $k=1,2,\ldots,N-1$, are orthogonal to υ_N, the following property holds:

$$\begin{aligned}&\sum_{j=1}^N\upsilon_{jk}=0,\quad k=1,2,\ldots,N-1,\\ &\upsilon_N=\frac{1}{\sqrt{N}}[1\ 1\ \ldots\ 1]^{\mathrm{T}}.\end{aligned} \quad (51)$$

Therefore, we have:

$$V^{-1}QV=-D_Q \quad (52)$$

where D_Q is a diagonal matrix with the diagonal entry μ_i, $i=1,2,\ldots,N$.

Due to property (51), the matrix V transforms the all 1's matrix Θ to a diagonal one as well:

$$\begin{aligned}V^{-1}\Theta V&=\left[(V^{-1}\Theta V)_{ik}\right]\\ &=\left[\left(\sum_{j=1}^N\upsilon_{ji}\right)\left(\sum_{j=1}^N\upsilon_{jk}\right)\right]\\ &=D_\Theta\end{aligned} \quad (53)$$

where D_Θ is a diagonal matrix with diagonal entry $(D_\Theta)_{ii}=0$, $i=1,2,\ldots,N-1$, and $(D_\Theta)_{NN}=N$. Choose the transformation matrix as follows:

$$T=V\otimes I_2. \quad (54)$$

We have:

$$\begin{aligned}T^{-1}GT&=(V\otimes I_2)^{-1}\left(I_N\otimes\begin{bmatrix}0&1\\-\cos\upsilon_{\text{target}}t&-\gamma\end{bmatrix}\right.\\ &\quad\left.+Q\otimes\begin{bmatrix}0&0\\\kappa&0\end{bmatrix}+\Theta\otimes\begin{bmatrix}0&0\\-\bar{k}_1&-\bar{k}_2\end{bmatrix}\right)(V\otimes I_2)\\ &=I_N\otimes\begin{bmatrix}0&1\\-\cos\upsilon_{\text{target}}t&-\gamma\end{bmatrix}-D_Q\otimes\begin{bmatrix}0&0\\\kappa&0\end{bmatrix}\\ &\quad+D_\Theta\otimes\begin{bmatrix}0&0\\-\bar{k}_1&-\bar{k}_2\end{bmatrix}\\ &=\mathrm{diag}\{C_i\},\end{aligned} \quad (55)$$

where

$$C_i = \begin{cases} \begin{bmatrix} 0 & 1 \\ -\cos v_{\text{target}}t - \mu_i\kappa & -\gamma \end{bmatrix}, & i = 1, 2, \dots, N-1, \\ \begin{bmatrix} 0 & 1 \\ -\cos v_{\text{target}}t - k_1 & -k_2 - \gamma \end{bmatrix}, & i = N. \end{cases} \quad (56)$$

This completes the proof of the lemma. □

We are now in the position to state the main theorem of this subsection.

Theorem 3. *For system parameters κ and γ that satisfy*

$$\kappa > \frac{1}{\mu_{N-1}}, \qquad \gamma > \frac{v_{\text{target}}}{2(\mu_{N-1}\kappa - 1)}, \quad (57)$$

where μ_{N-1} is the second smallest eigenvalue of the matrix $(-Q)$, choose the control parameters

$$k_1 > 1, \qquad k_2 > \max\left\{\frac{v_{\text{target}}}{2k_1} - \gamma, 0\right\}, \quad (58)$$

then the error system for individual particles (42) is locally asymptotically stable.

Proof of Theorem 3. We use the classic Lyapunov theory to prove the local stability of the error system (42).

From Lemma 4, under similarity transformation $z = T^{-1}e$, system (44) is transferred to the following one:

$$\dot{z} = \text{diag}\{C_i\}z, \quad (59)$$

where C_i is represented in (56).

Define the following Lyapunov function candidate:

$$W(t, z) = \sum_{i=1}^{N}\left\{\frac{\varepsilon_i}{2}z_{i1}^2 + \frac{1}{2}(\lambda_i z_{i1} + z_{i2})^2\right\} + \sum_{i=1}^{N}\left\{\frac{1}{2}[1 + \cos(v_{\text{target}}t)]z_{i1}^2\right\}, \quad (60)$$

where ε_i, and λ_i, $i = 1, \dots, N$, are design parameters.

We can see that

$$W_1(z) \le W(t, z) \le W_2(z), \quad (61)$$

where $W_1(z)$ and $W_2(z)$ are both positive definite:

$$W_1(z) = \sum_{i=1}^{N}\left\{\frac{\varepsilon_i}{2}z_{i1}^2 + \frac{1}{2}(\lambda_i z_{i1} + z_{i2})^2\right\}$$

$$W_2(z) = \sum_{i=1}^{N}\left\{\left(1 + \frac{\varepsilon_i}{2}\right)z_{i1}^2 + \frac{1}{2}(\lambda_i z_{i1} + z_{i2})^2\right\}. \quad (62)$$

Take the time derivative of $W(t, z)$ along the system dynamics (59). We have:

$$\begin{aligned} \dot{W}(t, z) = &\sum_{i=1}^{N-1}[\varepsilon_i + \lambda_i(\lambda_i - \gamma) - \mu_i\kappa + 1]z_{i1}z_{i2} \\ &+ \sum_{i=1}^{N-1}\left[-\frac{1}{2}v_{\text{target}}\sin(v_{\text{target}}t) - \lambda_i\cos(v_{\text{target}}t) - \lambda_i\mu_i\kappa\right]z_{i1}^2 \\ &+ \sum_{i=1}^{N-1}(\lambda_i - \gamma)z_{i2}^2 + \frac{1}{2}[\varepsilon_N + \lambda_N^2 - \lambda_N(k_2 + \gamma) - k_1 + 1]z_{N1}z_{N2} \\ &- \lambda_N k_1 z_{N1}^2 - (k_2 + \gamma - \lambda_N)z_{N2}^2 \\ &- \frac{1}{2}\sin(v_{\text{target}}t)v_{\text{target}}z_{N1}^2. \end{aligned} \quad (63)$$

Fig. 2. Local stability of the equilibrium points $(\phi_1, \dot{\phi}_1, \phi_2, \dot{\phi}_2, \phi_3, \dot{\phi}_3) = (0.1941, 0, 0.9360, 0, 4.7747, 0)$ in the presence of linear particle interactions. (a) Particle positions; (b) Particle velocities.

Under the condition on κ:

$$\kappa > \frac{1}{\min\limits_{i \le N-1}(\mu_i)} = \frac{1}{\mu_{N-1}}, \quad (64)$$

we have $\mu_i\kappa > 1$. Choose the design and control parameters:

$$\lambda_i < \gamma, \quad (65)$$

$$\varepsilon_i = \lambda_i(\gamma - \lambda_i) + \mu_i\kappa - 1 > 0, \quad i = 1, 2, \dots, N-1 \quad (66)$$

$$\lambda_N < \gamma + k_2, \quad (67)$$

$$\varepsilon_N = \lambda_N(k_2 + \gamma - \lambda_N) + k_1 - 1 > 0, \quad (68)$$

$$k_1 > 1, \quad (69)$$

so that the cross terms are zero. Bounding the sinusoidal terms, we obtain

$$\begin{aligned} \dot{W}(t, z) \le &-\sum_{i=1}^{N-1}\left(\lambda_i\mu_i\kappa - \frac{1}{2}v_{\text{target}} - \lambda_i\right)z_{i1}^2 \\ &- \sum_{i=1}^{N-1}(\gamma - \lambda_i)z_{i2}^2 - \left(\lambda_N k_1 - \frac{1}{2}v_{\text{target}}\right)z_{N1}^2 \\ &- (k_2 + \gamma - \lambda_N)z_{N2}^2. \end{aligned} \quad (70)$$

Because of the following condition on γ:

$$\gamma > \frac{v_{\text{target}}}{2\min\limits_{i \le N-1}(\mu_i\kappa - 1)} = \frac{v_{\text{target}}}{2(\mu_{N-1}\kappa - 1)}, \quad (71)$$

there exists $\lambda_i < \gamma$, $i = 1, \dots, N-1$, such that

$$\lambda_i\mu_i\kappa - \frac{1}{2}v_{\text{target}} - \lambda_i > 0. \quad (72)$$

To get $\lambda_N k_1 - \frac{1}{2}v_{\text{target}} > 0$, we need $\lambda_N > v_{\text{target}}/(2k_1)$. Combining with (67), we need to choose the following control parameter so