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Consensus on scale-free network

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***Abstract*— In this paper, we study the consensus on scale-free network, which is known as more robust and immune to the random mutation and perturbation than (binomial) random network. The underlying dynamics of system is derived from a discrete random process by averaging the state of agents weighted by agent degree. In such framework, we address the consensus on different categories, on fixed/switching topology, on undirected/directed graph based on martingale convergence theorem.**

***Index Terms*— Network control system, consensus problem, complex network, scale-free network, stochastic stability**

I. Introduction

The consensus problem in coupled dynamical networks and systems has been studied for years. As the cooperative multi-agent system has been proved to improve the overall flexibility and effectiveness of the system, the system coordination requires that individual agents share a consistent information of the world [1]. The information consensus lays the ground work to ensure the system achieving cooperative behavior effectively.

Information consensus is studied from both deterministic and probabilistic domains. Olfati-Saber and Murray in [2] address the consensus problem under a variety of deterministic assumptions, from directed networks with both fixed and switching topology, to undirected network with communication time-delays, using the approach of algebra graph theory and matrix theory. Jadbabaie etc. in [3] propose a deterministic consensus algorithm through averaging the neighbor states using the approach of ergodic theorem under different scenarios: discrete leaderless coordination, discrete leader-following coordination, as well as leader following in continuous time. On the probabilistic side, Hatano and Mesbahi in [5] consider the agreement over an undirected random network using Lyapunov methods from stochastic control theory. In [4], Wu considers the dynamics convergence problem in probability over binomial random directed network using inhomogeneous Markov chain, matrix theory and graph theory.

Scale-free network is one of the most fascinating discovery in complex networks topology by the end of last century. Albert Barabási and Réka Albert proposed that both the world wide web and the actor collaboration networks have their nodes' degree following a power law distribution in 1999. Later on, a tremendous number of networks (eg. social network, epidemiology, neural network, computer network) are identified to have the same power laws distribution. What is more, this power laws distribution structure always holds independently from the scale of the network. In another word, the advantages from scale-free network can be applied to various applications flexibly without the constraints from the number of the nodes in the network. Thus, the network is called scale-free network, which is a non-uniform random network. The scale-free network structure is a universal feature extracted from the natural, social, biological, and man-made network systems which are always robust against self mutation and environmental perturbation. Therefore, it would be fascinating to embed such robust and prominent feature from scale-free network framework into the multi-agent system to improve its cooperative behavior.

We address the consensus problem of decentralized dynamic system on scale-free network, which has turned out to be a robust and extendable communicating/sensing network topology. To our best knowledge, it is the first rigorous consensus proof on scale-free network. Based on the proof, we can apply such communication structure into a lot of multi-agent applications to improve system robustness, such like autonomous robotic system, transportation control system, mobile sensor network, etc. For example, given a group of agents assigned to achieve some cooperative tasks, we can build up the underlying communication topology network according to the scale-free network mechanism — preferential attachment. Based on our theoretical work, the system is mathematically proved to reach consensus, which provides solid basis for the system to achieve meaningful applications. Furthermore, it has been addressed that such multi-agent system can perform robustly to random link dropout or agent disfunction.

The outline of this paper is as follows. In Section II, we introduce the basic notions and terminologies in graph theory, the scale-free network model and its properties. In Section III, we presents a coordinated stochastic dynamical system based on the scale-free network mechanism. Then we

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prove the stochastic consensus stability of a directed finite scale-free network dynamical system in Section IV using the martingale convergence theorem. Later on, we generalize consensus result on both undirected/directed network. Finally this paper ends up with an overall conclusion as in Section V.

II. Scale-Free Network

A. Graph theory preliminary

We define *graph* G as a collection of vertices and edges $G = \{V(G), E(G)\}$ where $V(G)$ is a set of *vertices* $V(G) = [v_1\ v_2 \dots v_n]$ and $E(G)$ is a set of *edges* between the vertices $E(G) = \{(v_i,\ v_j)|v_i, v_j \in V(G)\}$. The edges of graphs may be imbued with direction. We define a graph to be *undirected* if every edge in the graph is an unordered pair between two distinct vertices. A graph is *directed* if the edge in the graph is an ordered pair of distinct vertices. Thus, in a directed graph, an edge has two distinct ends: a head and a tail. For an undirected graph, the *degree* of a vertex is the number of edges incident to the vertex. In a directed graph, each end is counted separately. The sum of head endpoints count toward the in-degree and the sum of tail endpoints count toward the out-degree. If there is a path between any two vertices of a graph, then graph is connected.

A *subgraph* of a graph G is a graph H such that: $V(G) \supseteq V(H), E(G) \supseteq E(H)$. A subgraph of X that is maximal, subject to being connected, is called a giant component (connected component). A graph (either undirected or directed) together with a function which assigns a positive real number to each edge is known as a *network*. we will take the default function as a scalar 1.

In term of finite and infinite element, a graph is *infinite* if it has infinitely many vertices or edges or both; otherwise the graph is *finite*. An infinite graph where every vertex has finite degree is called *locally finite*. We discuss finite graph in this paper.

In term of deterministic and probabilistic dependencies, a deterministic graph is with no facilities to accommodate probabilistic dependencies. A random (probabilistic) graph is a graph that is generated by random process. The theory of random graphs lies at the intersection between traditional graph theory and probability theory and studies the properties of typical random graphs. Distinction between these two dependencies will be further illustrated in III.

B. Network Models

Scale-free network is usually with the property that its degree follows a power-law distribution.

$$P[k] \sim k^{-\gamma} \tag{1}$$

in which, k is an integer denoting the node degree, $P[k]$ is the probability that a node connects with k other nodes. γ is a scalar coefficient, which usually ranges in

$$\gamma \in (2, \infty) \tag{2}$$

to ensure the expected value of degree k exists. The power law distribution denotes that some nodes has high degrees although most nodes are of low degree in a scale-free network. It is noteworthy that in Equation (1) it is to take the scale-free network as an undirected graph, which can be extended into a directed case by considering each vertex in undirected graph as an ordered pair of distinct vertices.

According to the statistic data from the directed scale-free network in the real world, both the in-degree and out-degree distribution follow power law approximately[11].

$$\begin{aligned} P\{k_{in}\} &\sim k_{in}^{-\gamma_{in}} \\ P\{k_{out}\} &\sim k_{out}^{-\gamma_{out}} \end{aligned} \tag{3}$$

To give a mechanistic explanation for such emergence of power laws, Bar*á*basi and R*é*ka introduced the preferential attachment model, in which the network grows by the addition of a single vertex at each time-step, with edges connected to it. The other end of each edge is connected to the other vertices already in the network, chosen at random with probability proportional to degree[15]. Let i denote the newly added vertex, N_i be the neighborhood of the newly added vertex, k_j be the degree of the vertex j, Then, the probability π_{ij} that a new vertex i will be connected to vertex j depends on the degree of vertex j as:

$$\pi_{ij} = \frac{k_j}{\sum_{j \in N_i} k_j} \tag{4}$$

C. Network Connectivity

Physics people have studied the connectivity of locally finite scale-free network through percolation theory, which entirely based on probabilistic arguments. Percolation theory is applicable to the connectivity of a system that contains so many particles that surface effects may be ignored and the system replaced by a model with an infinite number of particles in an unbounded volume. And the basic idea of percolation theory is to assume the probability that a given point belongs to an infinite cluster is known as the percolation probability, and a critical probability exists that the system phase transits from a non-percolating state to a percolating state[14].

Cohen and Erez refer to that in [12], if the power of degree distribution $\gamma \leq 3$, the critical probability threshold for a scale-free network system integrity being compromised is 1. In another word, an infinite scale-free network has a giant component almost surely under the condition $\gamma \leq 3$. The results ensure that connectivity of undirected scale-free network if $\gamma \leq 3$, and for directed scale-free network $\gamma_{in} \leq 3$ or $\gamma_{out} \leq 3$.

D. Why scale-free network

The scale-free network has many predominant advantages which can be used to improve the cooperative performance in a multi-agent system. First of all, it is known as a robust topology which is immune to random errors such as random removal of a rather large ratio of edges and vertices. Thus, for a cooperative control system, the scale-free network offers a reliable topology to ensure the information consensus under the condition certain modules are removed.

More, based on the "preferential attachment" mechanics, scale-free network offers the compatibility and flexibility adding new modules (edges and vertices) into the network. For a cooperative control system, it validates the addition of any vehicle without redesigning the control law. Such "plug and play" feature may improve the implementation of complex, distributed control system substantially in nowadays cooperation applications which are always embedded in information-rich environment[7].

Another advantage of the scale-free network is that it can be applied to very large scale system. As the scale of the cooperative system arises to a large number, the efficiency, feasibility and redundancy of the communication topology are being challenged. The scale-free network is a mathematic model extracted from the real-world complex network, so it is competent to be taken as the cooperative system topology with a rather large number of agents.

III. Problem Setup

Let (S, Σ, μ) be a measure space, where S is a set, Σ is a σ-algebra on S, and μ is a probability measure function defined on a σ-algebra Σ over a set S with values in $[0, 1]$.

We define the dynamics as a discrete-time stochastic system based on the buildup mechanism of scale-free network. For agent i, its state at time $t + 1$, can be updated by averaging the states of the agents around i weighted by agent degree at time t and itself . And such control law is based on the probability measure function in equation (3).

$$x_i[t+1] = \frac{1}{k_i + \sum\limits_{j \in N_i} k_j} \left(x_i[t] \cdot k_i + \sum_{j \in N_i} x_j[t] \cdot k_j \right) \quad (5)$$

in which $t \in \{1, 2, 3, \dots\}$, $x_i[t] \in R$ denotes the state of agent i at time t, k_i and k_j denote the degree of agent i and j respectively, and N_i denotes the neighborhood of agent i. Note that the concepts of "neighborhood" in deterministic and probabilistic domains are slightly different. In deterministic graph theory, the physical meaning of "j is in the neighborhood of i" is that there is an edge between i and j, and i and j communicate with each other; but in the probabilistic case, the concept of "neighborhood" is described in probability, j is in the neighborhood of i only if i and j are adjacent to each other, and the probability that an edge exists between i and j is non-zero, however, i and j do not definitely communicate to each other. In this paper, we will consider the probabilistic case only.

In order to extend dynamics on directed graph, we need the notation $j \in N_i^{out}$ which means j is in the neighborhood of i that i leads to, $i \rightarrow j$; vice versa, $j \in N_i^{in}$ denotes j is in the neighborhood of i, leading to i, $j \rightarrow i$. Then, one has

$$x_i[t+1] = \frac{x_i[t] \cdot k_i^{out} + \sum_{j \in N_i^{in}} x_j[t] \cdot k_j^{out}}{k_i^{out} + \sum_{j \in N_i^{in}} k_j^{out}} \quad (6)$$

or

$$x_i[t+1] = \frac{x_i[t] \cdot k_i^{in} + \sum_{j \in N_i^{in}} x_j[t] \cdot k_j^{in}}{k_i^{in} + \sum_{j \in N_i^{in}} k_j^{in}} \quad (7)$$

which means in directed graph, the state i is updated by how the in-degree neighborhood effects others and how agent i effects the others it leads to, and vice versa.

In simplicity, this system can be concisely represented as

$$\mathbf{x}[t+1] = M[t] \cdot \mathbf{x}[t] \quad (8)$$

where $M(t)$ is a stochastic matrix, in which

$$\begin{cases} M_{ij} = \dfrac{k_i^{in/out}}{k_i^{in/out} + \sum_{j \in N_i^{in}} k_j^{in/out}}, & i{=}j; \quad (9a) \\ M_{ij} = \dfrac{k_j^{in/out}}{k_i^{in/out} + \sum_{j \in N_i^{in}} k_j^{in/out}}, & j{\in}N_i; \quad (9b) \\ M_{ij} = 0, & \text{o.w.} \quad (9c) \end{cases}$$

Note that, since the dynamics is a random process, states stand for the expectation value over the degree probability distribution $M(t)$. Due to the randomness, the dynamical system network topology can be either initially randomly chosen due to preferential attachment rule (called fixed topology); or randomly updating due to preferential attachment rule from time to time (called switching topology). Both the fixed topology or the switching topology can be described by the same stochastic dynamic framework. It is different from the deterministic domain, in which fixed topology and switching topology are considered as two separate scenarios.

IV. Consensus over Scale-free network

Consensus is a common phenomenon in our daily life, from the computer virus spreading through internet, to birds and fish sharing information for the flocking and schooling motions. For multi-agent coordination, the consensus problem is a basic issue, which guarantees all the agents in system share information and be able to achieve state agreement to achieve some functional behavior. Based on the stochastic dynamics setup proposed in section III, we can develop the definition of consensus as follows.

Definition 1: The system reaches consensus almost surely (with probability one) when the state sequence $\mathbf{x}[t]$ converges to a steady state $\mathbf{x_{ss}}$ almost surely (a.s.) or with probability 1 (w.p.1), as $t \to \infty$. This is written in the form

$$\lim_{t\to\infty} P\{\mathbf{x}[t] \to \mathbf{x_{ss}}\} = 1 \tag{10}$$

Since the dynamics is a random process, which is different from the conventional deterministic stability problems, we employ the martingale convergence theorems to address the convergent stability with probability. First, we introduce the definition of martingale and supermartingale as follows.

Definition 2: ([9] pp.25) Let $y_n, n = 1, \ldots$ be a sequence of random variables, $B_n, n = 1, \ldots$ a nondecreasing sequence of σ-algebras, and y_n is measurable with respect to B_n. If y_n and satisfies $E[y_{n+1}|B_n] = y_n$ w.p.1, then the sequence $\{y_n, B_n\}$ is a *martingale*. If $E[y_{n+1}|B_n] \le y_n$ w.p.1, then the sequence $\{y_n, B_n\}$ is a *supermartingale*.

We draw the following martingale convergence theorem.

Lemma 1: ([10] pp. 31) If $\mathbf{x} = [\mathbf{x_1}, \mathbf{x_2}, \ldots]$ is a non-negative supermartingale sequence and $E|\mathbf{x_k}| < \infty$, then $\mathbf{x}$ converges w.p.1 to a random variable, as $k \to \infty$.

The main result of this section is as follows.

Theorem 1: For a finite directed scale-free network with dynamics as in equation (6), given that the average in-degree is greater than 1, the component in which one node connects to any other nodes with non-zero probability reaches consensus w.p.1.

Proof: We set the dynamic state sequence at time k as $\mathbf{x_k} = [x_1(k), x_2(k), \ldots]$.

Let stochastic function $S(\mathbf{x_k})$ as

$$S(\mathbf{x_k}) = \sum_i S_i(\mathbf{x_k}) = \sum_i \sum_{j\in N_i} \|x_i(k) - x_j(k)\| \ge 0 \tag{11}$$

It is trivial to consider the consensus convergence if $S(\mathbf{x_k})$ is a supermartingale and converges to 0 w.p.1. We will follow these two steps to prove the system consensus.

We set directed scale-free network equivalent model via following equivalent laws.

(1) We take the average in-degree as scalar δ_{in}, average out-degree as δ_{out}. Due to equation (3), it is clear that

$$\begin{aligned}\delta_{in} &= E\left[k_{in}\right] = \sum_{k_{in}=1}^{\infty} k_{in} \cdot P[k_{in}] \\ &= \sum_{k=1}^{\infty} k_{in} \cdot \alpha k_{in}^{-\gamma_{in}} = \sum_{k_{in}=1}^{\infty} \alpha k_{in}^{1-\gamma_{in}} \end{aligned} \tag{12}$$

in which α is a scalar to assure $\sum_{k_{in}=1}^{\infty} \alpha k_{in}^{-\gamma_{in}} = 1$. Note that k_{in} in equation (11) stands for the node degree and we will take d_i^{in} as the node i in-degree in the following, as well as d_i^{out} as the node i out-degree.

(2) We take the average state difference between any adjacent agents i and j as $\triangle x = \pm|\overline{x_i(k) - x_j(k)}|$.

Based on such equivalent model, we will prove the stochastic function $S(\mathbf{x_k})$ is a supermartingale based on system dynamics in equation (6).

$$\begin{aligned} &S_i(\mathbf{x_{k+1}}|\mathbf{x_k}) \\ &= \sum_{j\in N_i^{in}} \left\| \frac{\sum_{j\in N_i^{in}} x_j(k)\cdot d_j^{out} + x_i(k)\cdot d_i^{out}}{\sum_{j\in N_i^{in}} d_j^{out} + d_i^{out}} \right. \\ &\qquad \left. - \frac{\sum_{q\in N_j^{in}} x_q(k)\cdot d_q^{out} + x_j(k)\cdot d_j^{out}}{\sum_{q\in N_j^{in}} d_q^{out} + d_j^{out}} \right\| \\ &= \sum_{j\in N_i^{in}} \left\| \frac{\delta_{out}\cdot x_i(k)}{\delta_{out}\cdot\delta_{in} + \delta_{out}} - \frac{\delta_{out}\cdot x_j(k)}{\delta_{out}\cdot\delta_{in} + \delta_{out}} \right. \\ &\qquad \left. + \frac{\delta_{out}\cdot \sum_{j\in N_i^{in}} x_j(k)}{\delta_{out}\cdot\delta_{in} + \delta_{out}} - \frac{\delta_{out}\cdot \sum_{q\in N_j^{in}} x_q(k)}{\delta_{out}\cdot\delta_{in} + \delta_{out}} \right\| \\ &= \sum_{j\in N_i^{in}} \left\| \frac{x_i(k) - x_j(k)}{\delta_{in}+1} + \frac{\sum_{j\in N_i^{in}} x_j(k) - \sum_{q\in N_j^{in}} x_q(k)}{\delta_{in}+1} \right\| \\ &= \sum_{j\in N_i^{in}} \left\| \frac{x_i(k) - x_j(k) + \sum_{q\in N_j^{in}} (x_j(k) - x_q(k))}{\delta_{in}+1} \right\| \end{aligned}$$

Then, we can derive that

$$\max S_i(\mathbf{x_{k+1}}|\mathbf{x_k}) = \sum_{j\in N_i^{in}} \|\triangle x\| \tag{13}$$

$$\min S_i(\mathbf{x_{k+1}}|\mathbf{x_k}) = 0 \tag{14}$$

$$\min S_i(\mathbf{x_{k+1}}|\mathbf{x_k}) \le E\left[S_i(\mathbf{x_{k+1}}|\mathbf{x_k})\right] \le \min S_i(\mathbf{x_{k+1}}|\mathbf{x_k}) \tag{15}$$

In equation (13), the $S_i(\mathbf{x_{k+1}}|\mathbf{x_k})$ reaches maximum value, when the ith agent state $x_i(k)$ is always beyond (or below) all of its in-degree neighbor $x_j(k)$, and for every neighbor j, its state $x_j(k)$ is always beyond (or below) all of its in-degree neighbor $x_q(k)$. That is, $\forall j \in N_i^{in}, x_i(k) - x_j(k) > 0, \forall q \in N_j^{in}, x_j(k) - x_q(k) > 0$; or $\forall j \in N_i^{in}$, or $x_i(k) - x_j(k) < 0, \forall q \in N_j^{in}, x_j(k) - x_q(k) < 0$.

In equation (14), the $S_i(\mathbf{x_{k+1}}|\mathbf{x_k})$ reaches minimum value, when the ith agent state $x_i(k)$ is beyond (or below) its neighbor $x_j(k)$, and among the δ_{in} in-degree neighbors of j, $\frac{\delta_{in}-1}{2}$ items are below (or beyond) $x_j(k)$, $\frac{\delta_{in}+1}{2}$ items are beyond (or below) $x_j(k)$.

Since $\delta_{in} > 1$ and (15), we assume there exists a scalar μ that meets

$$E\left[S(\mathbf{x_{k+1}})|\mathbf{x_k}\right] = \mu \cdot \sum_{i} \sum_{j \in N_i^{in}} \|\triangle x\|, 0 \le \mu \le 1 \quad (16)$$

Obviously to see that

$$E\left[S(\mathbf{x_{k+1}})|\mathbf{x_k}\right] - S(\mathbf{x_k}) \le 0 \quad (17)$$

So $S(\mathbf{x_k})$ is a supermartingale. According to Lemma 1, $S(\mathbf{x_\infty})$ converges w.p.1 to a random variable. Next, we will prove that the expectation of $S(\mathbf{x_\infty})$ converges to zero.

First, note that when $\mu = 1$ or $\mu = 0$, $\min S_i(\mathbf{x_{k+1}}|\mathbf{x_k}) = \max S_i(\mathbf{x_{k+1}}|\mathbf{x_k}) = 0$ so that $\sum_i \sum_{j \in N_i^{in}} \|\triangle x\| = 0$, in other sense, the system reaches consensus.

More, consider when $0 < \mu < 1$, from (16)

$$E\left[S(\mathbf{x_{k+1}})|\mathbf{x_k}\right] = \mu \cdot S(\mathbf{x_k})$$

Taking expected value on both sides and iterate $E\left[S(\mathbf{x_k})\right]$, yields

$$\lim_{k\to\infty} E\left[S(\mathbf{x_k})\right] = E\left[S(\mathbf{x_0})\right] \lim_{k\to\infty} \prod_{1}^{k} \mu$$

Since

$$\prod_{1}^{k} \mu \le \exp\left(-\sum_{1}^{k}(1-\mu)\right)$$

$$\begin{aligned}\lim_{k\to\infty} E\left[S(\mathbf{x_k})\right] &\le E\left[S(\mathbf{x_0})\right] \lim_{k\to\infty} \exp\left(-\sum_{1}^{k}(1-\mu)\right) \\ &= E\left[S(\mathbf{x_0})\right] \exp(-\infty) = 0\end{aligned}$$

Therefore, $S(\mathbf{x_\infty})$ converges to zero w.p.1. ■

Remark 1: The proof above is based on the system dynamics in equation (6), by considering the neighborhood in-degree. It is simple to extend such results to the system dynamics in equation (7) regarding to the neighborhood out-degree as long as the average out-degree δ_{in} is greater than one.

Remark 2: The consensus can be applied to undirected network by taking an unordered edge in undirected network as a ordered pair of edges, as long as the average degree is assured, $\delta > 1$.

Remark 3: The component in which any two agents communicate in probability will reach consensus. It is a much more released requirement on the communication topology assumptions comparing to the previous deterministic results.

V. CONCLUSION

In this paper, we have proven the stochastic consensus over a robust and prominent network topology, namely, finite scale-free network. Based on "the averaging neighbor states weighted by neighbor degree" law, we set up a stochastic discrete-time dynamical system; and base on martingale convergence theorem, we proved the consensus of the dynamical system over a directed scale-free network. And later on, we applied the consensus results from directed graph to undirected graph.

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