Abstract

In this paper, we design an $H_\infty$ controller for a class of lower-triangular time-delay systems. Backstepping is applied to construct an explicit feedback controller, and the closed-loop system maintains internal stability and an $L_2$-gain from the disturbance input to the output. The design is delay-dependent. Simulations on an example system demonstrate the good performance of the proposed design. © 2002 Elsevier Science B.V. All rights reserved.

Keywords: Time-delay systems; $H_\infty$ control; Backstepping; Dissipation inequality; Delay-dependent

1. Introduction

Time delay occurs in various engineering systems. Since the existence of delay is often a source of instability and it greatly complicates the control system design, the study on time-delay systems has received considerable attentions from both classic control and process control communities, refer to [1–4,8,9,11,12,14–16,17–22]. As pointed out in [21], various type of approaches such as predictive control theory, state-space approach based on Hilbert space, optimal control, Lyapunov-type stability analysis and Razumikhin-type theory has been established and applied for the delay systems. Although results have been obtained to test the stability of linear systems with delayed states, constructing feedback control laws for systems with delay in control to meet stability and certain performance requirement remains challenging. In this paper, we will design an $H_\infty$ controller for a class of structured systems with delay in control. The dissipativity control theory will be applied.

During the last decade, there were considerable publications on the stability criteria for systems in the form of $\dot{x}(t) = Ax(t) + Bx(t - t_0)$, refer to [1,15,19] and the references therein. Based on these results, robust controller design for uncertain linear systems with time-delay state terms were discussed in [2,20]. For linear systems with delay in control, a famous deadtime compensation method is the Smith predictor [17], which converts the design problem for a process with delay to one without delay by eliminating time delay from the characteristic equation of the closed-loop system. However, the Smith predictor is for open-loop stable systems and cannot handle unmeasurable disturbances. Fuller [4] investigated an optimal regulator problem for
processes with delay in control. More recently in [9], Smith predictor control was combined with input–output linearization to design a controller for nonlinear time-delay systems.

For systems with the presence of persistent disturbances, $H_\infty$ control has been widely used as it provides explicit performance index in the sense of $L_2$ gain. With the development of the backstepping technique [10], $H_\infty$ control for lower-triangular systems has been intensively discussed recently, see [13,6,7]. It was shown that by exploring the structural information of the systems, the complicated controller design problem could be simplified by a recursive design procedure.

$H_\infty$ control for systems with delay in input has been discussed in [16,8,9]. In [8], robust stabilization results were given for linear systems with delay in control against additive perturbations, where the $H_\infty$ control problem for a delay system was converted to a problem for a delay-free system. However, the application of the method is limited since the equivalent mapping from the input to the output is only valid under some underlying conditions of the augmented systems. Another interesting work in robust $H_\infty$ control of time-delay systems was reported recently in [3]. State-delayed systems were considered, a delay-dependent Lyapunov function was chosen and the linear matrix inequality (LMI) method was used to construct the control law.

In this paper, we consider the $H_\infty$ control problem for lower-triangular structural systems with delay in control. Backstepping is used as a design tool for constructing the energy function which is nonlinear and delay-dependent. A recursive design procedure is presented and the final control law is dependent on the delay constant $t_d$. The closed-loop system maintains an $L_2$ gain from the disturbance input to the output with internal stability. Naturally, the performance index, the $L_2$ gain, is delay-dependent as well. Advanced dissipativity control theory serves as the underlying theory. It is the first to apply backstepping technique with time-delay characteristics.

The rest of the paper is organized as follows: In Section 2, we give the system configuration and define the main problem concerned in the paper. Then in Section 3, a recursive controller design procedure is presented using backstepping and our main theorem is given. Section 4 presents an example to illustrate the control performance. Finally the paper is concluded in Section 5.

*Notations*: The notation used in this paper is standard. $|\cdot|$ denotes the usual Euclidean norm for vectors. We say that $z : (0, T) \rightarrow \mathbb{R}^k$ is in $L_2(0, T)$ if $\int_0^T |z(t)|^2 \, dt < \infty$. $\lambda_{\text{max}}(P)$ and $\lambda_{\text{min}}(P)$ denote the maximum and the minimum eigenvalue of any square matrix $P$.

2. Problem statement

We consider a class of systems in the following state-space form:

\[
\begin{align*}
\dot{z} &= Az + B(\zeta_1 + p_0 \omega), \\
\dot{\zeta}_1 &= \zeta_2 + p_1 \omega, \\
\dot{\zeta}_2 &= \zeta_3 + p_2 \omega, \\
& \vdots \\
\dot{\zeta}_n &= u(t - t_d) + p_n \omega, \\
y &= z_1,
\end{align*}
\]

(1)

where

\[
A = \begin{bmatrix}
0 & 1 & 0 \\
& \ddots & \ddots \\
0 & 0 & 1 \\
0 & 0 & \cdots & 0
\end{bmatrix}, \quad B = \begin{bmatrix}
0 \\
\vdots \\
0 \\
1
\end{bmatrix},
\]


\[ z = (z_1, \ldots, z_k) \in \mathbb{R}^k, \xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n, \begin{bmatrix} z \\ \xi \end{bmatrix}^T \text{ is the state vector, } u \in \mathbb{R} \text{ is the control input, } \omega \in \mathbb{R}^m \text{ is the disturbance input, } y \in \mathbb{R} \text{ is the to-be-controlled output; } p_i \ (i = 0, 1, \ldots, n) \text{ is a constant matrix with appropriate dimension, and } t_d \text{ is the time-delay constant.} \]

The problem discussed in this paper is defined as follows.

**H_\infty Control Problem.** Find a feedback control law \( u(t - t_d) = u(z(t - t_d), \xi(t - t_d)) \) for system (1), such that the closed-loop system is globally uniformly asymptotically stable (GUAS) when \( \omega = 0 \). And with the nonzero disturbance \( \omega \in L_2(0, T), \forall T \geq 0 \), the following dissipation inequality is satisfied:

\[
\int_0^T |y|^2 \, dt \leq \mu \int_0^T |w|^2 \, dt + \nu(z(\tilde{t}_0), \xi(\tilde{t}_0), \tilde{t}_0),
\]

where \( \mu \) is a positive constant, \( \nu \) is a positive semidefinite function and \( z(\tilde{t}_0), \xi(\tilde{t}_0) \), are the initial conditions where \( \tilde{t}_0 \in [-t_d, 0] \).

**Remark 1.** In the dissipation inequality (2), the effect of initial conditions is considered. Since \( z(\tilde{t}_0) \) and \( \xi(\tilde{t}_0) \) happened in history which cannot be compensated, it is reasonable to be included as initial conditions. This also explains the difference between the dissipation inequality for delayed systems and that for delay-free systems.

### 3. Main result

In this section, we firstly present a recursive controller design procedure by backstepping. Then we show that with the constructed controller, the \( H_\infty \) Control Problem is solved.

**Step 0:** We start by considering the \( z \)-subsystem with \( \xi_1 \) as the virtual control input. Choose the storage function

\[
V_0(z) = z^T P z,
\]

where \( P \) is a positive definite symmetric matrix solving the algebraic Riccati equation:

\[
A^T P + PA - 2 \varepsilon P B B^T P + Q = 0,
\]

where \( \varepsilon \) is a positive constant and \( Q \) is a positive definite symmetric matrix.

Differentiating (3) along the solution of \( z \)-subsystem, we have

\[
\dot{V}_0(z) = 2z^T PAz + 2z^T PB \xi_1 + 2z^T PB p_0 \omega
\]

\[
\leq 2z^T PAz + 2z^T PB \xi_1 + \frac{1}{\tau_0} |z^T PB p_0|^2 + \tau_0 |\omega|^2,
\]

where the inequality \( 2ab \leq a^2 + b^2 \ (a, b \in \mathbb{R}) \) is used in the second inequality, and \( \tau_0 \) is a positive design parameter.

If we let the virtual control \( \xi_1 = \xi_1^*(z) \) as

\[
\xi_1^*(z) = -B^T P z - \frac{1}{2 \tau_0} p_0 \Sigma_{10} B^T P z \triangleq \delta_{10} z
\]

then (5) turns to

\[
\dot{V}_0(z) \leq -z^T Q z + \tau_0 |\omega|^2.
\]
Step 1: Augment the \( z \)-subsystem with the \( \xi_1 \)-subsystem, and choose a storage function as

\[
V_1(z, \xi_1) = V_0(z) + (\xi_1 - \xi_1^*)^2. \tag{8}
\]

Denote

\[
\xi_1^*(z) = \delta_{10}z = \delta_{10}A z + \delta_{10}B \xi_1 + \delta_{10}B p_0 \omega
\]

\[
\triangle = \delta_{10}z + \delta_{11} \xi_1 + g_1 \omega. \tag{9}
\]

Differentiating \( V_1 \) along the solutions of the \((z, \xi_1)\)-subsystem yields

\[
\dot{V}_1(z, \xi_1) \leq -z^T Q z + \tau_0 |\omega|^2 + 2z^T PB(\xi_1 - \xi_1^*) + 2(\xi_1 - \xi_1^*)(\xi_2 + p_1 \omega - \xi_1^*)
\]

\[
= -z^T Q z + \tau_0 |\omega|^2 + 2(\xi_1 - \xi_1^*)(\xi_2 + z^T PB - \vartheta_{10}z - \vartheta_{11} \xi_1 - q_1 \omega + p_1 \omega)
\]

\[
\leq -z^T Q z + \tau_0 |\omega|^2 + 2(\xi_1 - \xi_1^*)(\xi_2 + z^T PB - \vartheta_{10}z - \vartheta_{11} \xi_1)
\]

\[
+ \frac{1}{\tau_1}|\xi_1 - \xi_1^*|^2|p_1 - q_1|^2 + \tau_1 |\omega|^2, \tag{10}
\]

where \( \tau_1 \) is a positive constant.

Choose the virtual control \( \xi_2^* = \xi_2^* \) as

\[
\xi_2^*(z, \xi_1) = -B^T P z - c_1(\xi_1 - \xi_1^*) + \vartheta_{10}z + \vartheta_{11} \xi_1 - \frac{1}{\tau_1}(\xi_1 - \xi_1^*)|p_1 - q_1|^2
\]

\[
\triangle = \delta_{20}z + \delta_{21} \xi_1, \tag{11}
\]

where \( c_1 > 0 \).

Denote \( \xi_1 = \xi_1 - \xi_1^* \), then (10) turns to

\[
\dot{V}_1(z, \xi_1) \leq -z^T Q z - 2c_1 \xi_1^2 + (\tau_0 + \tau_1)|\omega|^2. \tag{12}
\]

Claim. Given any index \( 2 \leq l \leq n - 1 \), for the system

\[
\dot{z} = Az + B(\xi_1 + p_0 \omega),
\]

\[
\dot{\xi}_1 = \xi_2 + p_1 \omega,
\]

\[
\dot{\xi}_2 = \xi_3 + p_2 \omega,
\]

\[
\vdots
\]

\[
\dot{\xi}_l = \xi_{l+1} + p_{l+1} \omega, \tag{13}
\]

there exist \( l + 1 \) smooth functions

\[
\xi^*_i = \xi^*_i(z, \xi_1, \ldots, \xi_{i-1}) = \delta_{i0}z + \delta_{i1} \xi_1 + \cdots + \delta_{i,l-1} \xi_{i-1}, \quad 1 \leq i \leq l + 1
\]

such that in new coordinates

\[
\hat{\xi}_i = \xi_i - \xi^*_i(z, \xi_1, \ldots, \xi_{i-1}), \quad 1 \leq i \leq l
\]
the storage function
\[ V_l = V_0 + \sum_{i=1}^{l} \xi_i^2 \]  
has time derivative, with $\tilde{\xi}_{l+1} = \tilde{\xi}_{l+1}^*$, satisfying the dissipation inequality
\[ \dot{V}_l(z, \tilde{\xi}_l) \leq -z^TQz - \sum_{i=1}^{l} 2c_i \tilde{\xi}_i^2 + \sum_{i=1}^{l} \tau_i |\omega|^2. \]

The proof of the Claim is straightforward and is given in the appendix.

**Step n:** Let
\[ V_n(z, \tilde{\xi}_n) = V_{n-1} + (\tilde{\xi}_n - \tilde{\xi}_n^*)^2 + V_d(t, z, \tilde{\xi}), \]
where
\[ V_d(t, z, \tilde{\xi}) = \int_{-t_d}^{0} \int_{t_d + \theta}^{t} \left[ r_0 z^T(s)z(s) + \sum_{i=1}^{n} r_i \tilde{\xi}_i^T(s)\tilde{\xi}_i(s) + \kappa \omega(s)^T\omega(s) \right] ds d\theta \]
\[ + \int_{-t_d}^{0} \int_{t_d + \theta}^{t} \left[ l_0 z^T(s)z(s) + \sum_{i=1}^{n} l_i \tilde{\xi}_i^T(s)\tilde{\xi}_i(s) \right] ds d\theta \]
and $r_0, r_i, l_0, l_i, \kappa$ are positive design parameters.

From Step $n-1$ by the Claim, $\tilde{\xi}_n^* = \delta_n z + \delta_n \tilde{\xi}_1 + \cdots + \delta_{n-1} \tilde{\xi}_{n-1}$. Denote
\[ \tilde{\xi}_n(z, \tilde{\xi}_1, \ldots, \tilde{\xi}_{n-1}) = \left( \frac{\partial \tilde{\xi}_n^*}{\partial z} \tilde{\xi} + \frac{\partial \tilde{\xi}_n^*}{\partial \tilde{\xi}_1} \tilde{\xi}_1 + \cdots + \frac{\partial \tilde{\xi}_n^*}{\partial \tilde{\xi}_{n-1}} \tilde{\xi}_{n-1} \right) \]
\[ \triangleq \partial_{n0} z + \partial_{n1} \tilde{\xi}_1 + \cdots + \partial_{n(n-1)} \tilde{\xi}_{n-1} + \partial_{nn} \tilde{\xi}_n + \partial_{nn} \omega. \]

Differentiating $V_n$ along the solutions of system (1), we have
\[ \dot{V}_n \leq -z^TQz - \sum_{i=1}^{n-1} 2c_i \tilde{\xi}_i^2 + \sum_{i=1}^{n-1} \tau_i |\omega|^2 + 2 \tilde{\xi}_n z^T(t - t_d) + \tilde{\xi}_{n-1} - \tilde{\xi}_n^* \]
\[ + \dot{V}_d(t, z, \tilde{\xi}), \]
where $\tilde{\xi}_n = \tilde{\xi}_n - \tilde{\xi}_n^*$, and
\[ L_0 = -\delta_{n-1,0} - \partial_{n0}, \]
\[ L_i = \begin{cases} -\delta_{n-1,i} - \partial_{ni}, & 1 \leq i \leq n-2, \\ 1 - \partial_{ni}, & i = n-1, \\ -\partial_{ni}, & i = n. \end{cases}, \]

Note that
\[ 2 \tilde{\xi}_n(z, \tilde{\xi}_1, \ldots, \tilde{\xi}_{n-1}) \omega \leq \frac{1}{\tau_n} \tilde{\xi}_n^2 |p_n| |\omega_n|^2 + |\omega_n|^2. \]
Denote
\[
\begin{align*}
  u^*(z, \zeta) &= c_n \dot{z}_n + L_0 z + \sum_{i=1}^{n} L_i \dot{z}_i + \frac{1}{2} \dot{z}_n | p_n - q_n |^2 \\
  \Delta &= K_0 z + K_1 \dot{z}_1 + K_2 \dot{z}_2 + \cdots + K_n \dot{z}_n
\end{align*}
\]
then (20) turns to
\[
\begin{align*}
  \dot{Y}_n &\leq - z^T Q z - \sum_{i=1}^{n} 2c_i \dot{z}_i^2 + \sum_{i=1}^{n} \tau_i |\omega|^2 + 2 \dot{z}_n [u(t - t_d) + u^*(z, \zeta)] + \dot{V}_d(t, z, \zeta).
\end{align*}
\]
Since \(z(t), \xi_i(t), 1 \leq i \leq n\) are continuously differentiable for \(t \geq 0\), we have
\[
\begin{align*}
  z(t - t_d) &= z(t) - \int_{-t_d}^{0} \dot{z}(t + \theta) \, d\theta \\
  &= z(t) - \int_{-t_d}^{0} [A z(t + \theta) + B(\xi_1(t + \theta) + p_0 \omega(t + \theta))] \, d\theta, \\
  \xi_i(t - t_d) &= \xi_i(t) - \int_{-t_d}^{0} [\xi_{i+1}(t + \theta) + p_i \omega(t + \theta)] \, d\theta, \quad (1 \leq i \leq n - 1), \\
  \xi_n(t - t_d) &= \xi_n(t) - \int_{-t_d}^{0} [u(t - t_d + \theta) + p_n \omega(t + \theta)] \, d\theta.
\end{align*}
\]
Note that by using (25)–(27), the initial condition of system (1) shifts from \([-t_d, 0]\) to \([-2t_d, 0]\), while the system stability remains same [5, p. 131].
Now choose the true delayed control law
\[
u(t - t_d) = - K_0 z(t - t_d) - K_1 \dot{z}_1(t - t_d) - K_2 \dot{z}_2(t - t_d) - \cdots - K_n \ddot{z}_n(t - t_d).
\]
Substitute (25)–(27), (28) into (24),
\[
\begin{align*}
  \dot{V}_n &\leq - z^T Q z - \sum_{i=1}^{n} 2c_i \dot{z}_i^2 + \sum_{i=1}^{n} \tau_i |\omega|^2 + \eta_1(t, z, \zeta) + \eta_2(t, z, \zeta) + \eta_3(t, z, \zeta) + \dot{V}_d(t, z, \zeta),
\end{align*}
\]
where
\[
\begin{align*}
  \eta_1(t, z, \zeta) &= 2 \dot{z}_n \int_{-t_d}^{0} \left[ K_0 A z(t + \theta) + K_0 B \dot{z}_1(t + \theta) + \sum_{i=1}^{n-1} K_i \dot{z}_{i+1}(t + \theta) \right] \, d\theta, \\
  \eta_2(t, z, \zeta) &= 2 \dot{z}_n \int_{-t_d}^{0} \left[ K_n K_0 z(t - t_d + \theta) + \sum_{i=1}^{n} K_n K_i \dot{z}_i(t - t_d + \theta) \right] \, d\theta, \\
  \eta_3(t, z, \zeta) &= 2 \dot{z}_n \int_{-t_d}^{0} \left( K_0 p_0 + \sum_{i=1}^{n} K_i p_i \right) \omega(t + \theta) \, d\theta.
\end{align*}
\]
Using \(2ab \leq a^2 + b^2 (a, b \in \mathbb{R})\) again, we obtain
\[
\eta_1(t, z, \zeta) \leq t_d \left( \frac{1}{r_0} |K_0 A|^2 + \frac{1}{r_1} |K_0 B|^2 + \sum_{i=1}^{n-1} \frac{1}{r_{i+1}} |K_i|^2 \right) \dot{z}_n^2
\]
\[ + \int_{-t_0}^{0} \left[ r_0 \|z(t + \theta)\|^2 + \sum_{i=1}^{n} r_i \|\xi(t + \theta)\|^2 \right] d\theta, \]

\[ \eta_2(t, z, \xi) \leq t_d \left( \frac{1}{T_0} \|K_n K_0\|^2 + \sum_{i=1}^{n} \frac{1}{T_i} \|K_i\|^2 \right) \frac{\xi_i^2}{\xi_n^2} \]

\[ + \int_{-t_0}^{0} \left[ l_0 \|z(t - t_d + \theta)\|^2 + \sum_{i=1}^{n} l_i \|\xi(t - t_d + \theta)\|^2 \right] d\theta, \]

\[ \eta_3(t, z, \xi) \leq t_d \frac{1}{K} (K_0 p_0 + \sum_{i=1}^{n} K_i p_i) \xi_n^2 + \int_{-t_0}^{0} \|\omega(t + \theta)\|^2 d\theta. \quad (31) \]

Notice that

\[ \dot{V}_n(t, z, \xi) = t_d r_0 z^T z + \sum_{i=1}^{n} t_d \xi_i \xi_i^T \xi_i + t_d K\omega^T \omega \]

\[ - \int_{-t_0}^{0} \left[ r_0 z^T(t + \theta)z(t + \theta) + \sum_{i=1}^{n} r_i \xi_i^T(t + \theta)\xi_i(t + \theta) + \tau\omega(t + \theta)^T \omega(t + \theta) \right] d\theta \]

\[ + t_d l_0 z^T z + \sum_{i=1}^{n} t_d l_i \xi_i^T \xi_i \]

\[ - \int_{-t_0}^{0} \left[ l_0 z^T(t - t_d + \theta)z(t - t_d + \theta) + \sum_{i=1}^{n} l_i \xi_i^T(t - t_d + \theta)\xi_i(t - t_d + \theta) \right] d\theta. \quad (32) \]

Substitute (31), (32) into (29),

\[ \dot{V}_n \leq -z^T Qz - \sum_{i=1}^{n} 2c_i \xi_i^2 + \sum_{i=1}^{n} \tau_i \|\omega\|^2 \]

\[ + t_d (r_0 + l_0) \|z\|^2 + \sum_{i=1}^{n} t_d (r_i + l_i) \|\xi_i\|^2 + t_d \phi \xi_n^2 + t_d K \|\omega\|^2, \quad (33) \]

where

\[ \phi = \left( \frac{1}{r_0} \|K_0 A\|^2 + \frac{1}{r_1} \|K_0 B\|^2 + \sum_{i=1}^{n-1} \frac{1}{r_{i+1}} \|K_i\|^2 \right) \]

\[ + \left( \frac{1}{T_0} \|K_n K_0\|^2 + \sum_{i=1}^{n} \frac{1}{T_i} \|K_i\|^2 \right) + \frac{1}{K} (K_0 p_0 + \sum_{i=1}^{n} K_i p_i). \quad (34) \]

Since the mapping

\[ \Theta : (z, \xi_1, \xi_2, \ldots, \xi_n) \mapsto (z, \tilde{\xi}_1, \tilde{\xi}_2, \ldots, \tilde{\xi}_n) \]

is a linear transformation whose Jacobian matrix is lower triangular with all diagonal components equal to the constant one, it is easy to get

\[ \xi_i = \tilde{\xi}_i + \phi_{i-1} \tilde{\xi}_{i-1} + \cdots + \phi_1 \tilde{\xi}_1 + \phi_0 z, \quad (35) \]

where \( \phi_0 \) is a constant matrix, and \( \phi_1, \ldots, \phi_{i-1} \) are constants.
Applying
\[(a_1 + a_2 + \cdots + a_k)^2 \leq k(a_1^2 + a_2^2 + \cdots + a_k^2), \quad a_1, a_2, \ldots, a_k \in \mathbb{R}\]
to (35),
\[
\|\xi_i\|^2 \leq (i + 1)(|\varphi_0|^2|z|^2 + \varphi_1^2\xi_1^2 + \cdots + \varphi_{i-1}^2\xi_{i-1}^2 + \xi_i^2).
\]  
(36)
Therefore (33) turns to
\[
\dot{V}_n \leq -\alpha|z|^2 - \sum_{i=1}^{n} \beta_i \xi_i^2 + \tau|\omega|^2,
\]  
(37)
where
\[
\alpha = \lambda_{\text{min}}(Q) - t_d \left[ (r_0 + l_0) + \sum_{i=1}^{n} (r_i + l_i)(i + 1)|\varphi_0|^2 \right],
\]
\[
\beta_i = \begin{cases} 
2c_i - t_d(r_i + l_i)(i + 1)|\varphi_i|^2, & 1 \leq i \leq n - 1, \\
2c_i - t_d[(r_i + l_i)(i + 1) + \phi], & i = n,
\end{cases}
\]
\[
\tau = \sum_{i=1}^{n} \tau_i + t_d\kappa.
\]  
(38)
It can be seen that \(\alpha, \beta_i (1 \leq i \leq n), \tau\) are dependent on \(t_d\). For certain small \(t_d\), it is possible to find positive design parameters \(\alpha, Q, c_i, r_0, r_i, l_0, l_i, \tau_i, \kappa\) to ensure that \(\alpha, \beta_i > 0\).

Now we are in the position to state our main result.

**Theorem 1.** The feedback control law (28) solves the \(H_\infty\) Control Problem if there exist positive constants/ matrix \(\alpha, Q, c_i, r_0, r_i, l_0, l_i, \tau_i, \kappa\), \((1 \leq i \leq n)\) such that \(\alpha, \beta_i\) are positive as defined in (38).

The proof of the theorem is given in the appendix.

**Remark 2.** The controller design method proposed above is a recursive procedure. It follows the backstepping design where energy function and virtual control law are stepwisely constructed. The delay in the input makes the energy function more complicated than that for delay-free systems. It includes an integral term of the states within the delayed time period. The final dissipation inequality (37) shows the energy dissipating along the closed-loop trajectory. The design is delay-dependent.

**Remark 3.** The integral form of energy function (18) was firstly proposed in [12] and recently used in [3]. While in [3] state-delayed systems were discussed and initial conditions were assumed to be zero, the present paper deals with systems with delay in control and initial conditions are explicitly addressed in the dissipation inequality (2). It is also noted that the proving method in [3] is only valid for the \(L_2\) space defined on \([0, \infty)\) instead of that on \([0, T], \forall T > 0\).

4. An illustrated example

We consider the following lower-triangular system:
\[
\begin{align*}
\dot{z}_1 &= z_2, \\
\dot{z}_2 &= \xi + 0.2\omega, \\
\dot{\xi} &= u(t - t_d) + 0.5\omega.
\end{align*}
\]  
(39)
Following the design procedure described in Section 3, and choosing the design parameters 
\[ e = 20, \quad Q = I, \quad c_1 = 4, \quad \tau_0 = \tau_1 = 0.5, \]
we obtain the control law
\[ u(t - t_d) = -22.3741z_1(t - t_d) - 23.8378z_2(t - t_d) - 9.6352z_3(t - t_d). \]
Simulation results of the closed-loop system is shown in Figs. 1 and 2, where \( t_d = 0.1, \omega = 1, \) and the initial condition is \( [z_1 z_2 z_3]^T = [0.5 0.5 0.5]^T. \) Satisfactory performance for all states can be observed.
To compare our controller to the open-loop system and the standard LQ design method, we show in Fig. 3 the output trajectories under the three situations. It can be seen that the open-loop system is unstable; the LQ design method stabilizes the system but cannot attenuate the disturbance’s effect (as it was not considered in the design); and the $H\infty$ control design by the proposed method ensures the stability and achieves disturbance attenuation to the output.

5. Conclusions

The $H\infty$ control problem for the lower-triangular systems with delay in control has been studied. An explicit feedback control law has been built, and the closed-loop system maintains the $L2$-gain from the disturbance input to the output with internal stability. Backstepping technique was applied to the structural system. The design is delay-dependent. Simulation results on a third-order example system have demonstrated its good performance compared to the open-loop system and the standard LQ design method.

Acknowledgements

Financial support from the Australian Research Council (ARC) is gratefully acknowledged.

Appendix

Proof of the Claim. As stated previously, the Claim holds for system $l = 1$. Assume that the Claim is true for system $l = k - 1$, we wish to prove the claim for system $l = k$.

Consider the storage function

$$V_k = V_{k-1} + \xi_k^2.$$  \hspace{1cm} (A.1)
Differentiating it along the solutions of the \((z, \xi_1, \ldots, \xi_k)\)-subsystem in (14), we have
\[
\dot{V}_k \leq -z^T Q z - \sum_{i=1}^{k-1} 2 c_i \dot{\xi}_i^2 + \sum_{i=1}^{k-1} \tau_i |o|^2 + 2 \ddot{\xi}_k [\ddot{\xi}_{k-1} + \ddot{\xi}_{k+1} + p_k o - \dot{\xi}_k^*],
\]
\[(A.2)\]
where \(\ddot{\xi}_k = \ddot{\xi}_k - \dot{\xi}_k^*\).

Note that
\[
\dot{\xi}_k^* = \delta_{k0} \dot{z} + \delta_{k1} \dot{\xi}_1 + \cdots + \delta_{k,k-1} \dot{\xi}_{k-1}
\]
\[
= \Delta \theta_{k0} \dot{z} + \theta_{k1} \dot{\xi}_1 + \cdots + \theta_{kk} \dot{\xi}_k + \theta_k \omega.
\]
\[(A.3)\]
Substituting (A.3) into (A.2), and collecting similar terms, we get
\[
\dot{V}_k \leq -z^T Q z - \sum_{i=1}^{k-1} 2 c_i \dot{\xi}_i^2 + \sum_{i=1}^{k-1} \tau_i |o|^2 + 2 \dot{\xi}_k \left[ \dot{\xi}_{k+1} + \sigma_0 \dot{z} + \sum_{i=1}^{k} \sigma_i \dot{\xi}_i \right] + 2 \ddot{\xi}_k (p_k - \theta_k) \omega,
\]
\[(A.4)\]
where
\[
\sigma_0 = -\delta_{k-1,0} - \theta_{k0},
\]
\[
\sigma_i = \begin{cases} 
-\delta_{k-1,i} - \theta_{ki}, & 1 \leq i \leq k-2, \\
1 - \theta_{ki}, & i=k-1, \\
-\theta_{ki}, & i=k.
\end{cases}
\]
\[(A.5)\]
Note that the last term of (A.4) turns to
\[
2 \ddot{\xi}_k (p_k - \theta_k) \omega \leq \frac{1}{\tau_k} \frac{\ddot{\xi}_k^2}{|p_k + \theta_k|^2} + \frac{\tau_k |o|^2}{\tau_k},
\]
\[(A.6)\]
where \(\tau_k\) is a positive design parameter.

Choose the virtual control \(\dot{\xi}_{k+1} = \dot{\xi}_{k+1}^*\) as
\[
\dot{\xi}_{k+1} = -c_k \dot{\xi}_k - \sigma_0 \dot{z} - \sum_{i=1}^{k} \sigma_i \dot{\xi}_i - \frac{1}{2 \tau_k} \ddot{\xi}_k |p_k + \theta_k|^2
\]
\[
= \Delta \delta_{k+1,0} \dot{z} + \delta_{k+1,1} \dot{\xi}_1 + \cdots + \delta_{k+1,k} \dot{\xi}_k,
\]
\[(A.7)\]
where \(c_k\) is a positive design parameter, then we get
\[
\dot{V}_k \leq -z^T Q z - \sum_{i=1}^{k} 2 c_i \dot{\xi}_i^2 + \sum_{i=1}^{k} \tau_i |o|^2.
\]
\[(A.8)\]
This completes the proof of the Claim. □

**Proof of Theorem 1.** From the design procedure described in Section 3, we have the following Lyapunov function,
\[
V_n(t, z, \xi) = z^T P z + \sum_{i=1}^{n} \ddot{\xi}_i^2 + \int_{t_0}^{t} \int_{1+\theta} \left[ r_{0z}^T(z(s)z(s)) + \sum_{i=1}^{n} r_{i\xi_i}^T(z(s)\xi_i(s)) + \kappa o(s)^T o(s) \right] ds d\theta
\]
\[+ \int_{t_0}^{t} \int_{1-\theta} \left[ l_{0z}^T(z(s)z(s)) + \sum_{i=1}^{n} l_{i\xi_i}^T(z(s)\xi_i(s)) \right] ds d\theta.
\]
\[(A.9)\]
Due to (35), $V_n$ is positive definite in the coordinates $(z, \tilde{\xi})$. Its derivative along the closed-loop trajectory is

$$
\dot{V} \leq -z|z|^2 - \sum_{i=1}^{n} \beta_i \xi_i^2 + \tau |\omega|^2.
$$

(A.10)

If $z$, $\beta_i$, $\tau$ are all positive, when $\omega = 0$, $\dot{V}$ is negative definite. So the transferred states $(z, \tilde{\xi})$ is GUAS. Since (35), the original coordinates of system (1) is GUAS too.

When $\omega \neq 0$, taking the integral along time $t$ for both sides of (A.10) and simplifying, we obtain

$$
\int_0^T z|z|^2 \, dt \leq \tau \int_0^T |\omega|^2 \, dt + V_n(0,z(0),\tilde{\xi}(0)).
$$

(A.11)

Therefore,

$$
\int_0^T |y|^2 \, dt \leq \mu \int_0^T |w|^2 \, dt + \nu(z(\tilde{t}_0),\xi(\tilde{t}_0),\tilde{t}_0),
$$

(A.12)

where

$$
\mu = \tau/\alpha,
$$

$$
\nu(z(\tilde{t}_0),\xi(\tilde{t}_0),\tilde{t}_0) = V_d(0,z(0),\xi(0))/\alpha.
$$

(A.13)

This completes the proof of the theorem. □

References