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H_∞ control for a class of structured time-delay systems

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Abstract

In this paper, we design an H_∞ controller for a class of lower-triangular time-delay systems. Backstepping is applied to construct an explicit feedback controller, and the closed-loop system maintains internal stability and an L_2-gain from the disturbance input to the output. The design is delay-dependent. Simulations on an example system demonstrate the good performance of the proposed design. © 2002 Elsevier Science B.V. All rights reserved.

Keywords: Time-delay systems; H_∞ control; Backstepping; Dissipation inequality; Delay-dependent

1. Introduction

Time delay occurs in various engineering systems. Since the existence of delay is often a source of instability and it greatly complicates the control system design, the study on time-delay systems has received considerable attentions from both classic control and process control communities, refer to [1–4,8,9,11,12,14–16,17–22]. As pointed out in [21], various type of approaches such as predictive control theory, state-space approach based on Hilbert space, optimal control, Lyapunov-type stability analysis and Razumikhin-type theory has been established and applied for the delay systems. Although results have been obtained to test the stability of linear systems with delayed states, constructing feedback control laws for systems with delay in control to meet stability and certain performance requirement remains challenging. In this paper, we will design an H_∞ controller for a class of structured systems with delay in control. The dissipativity control theory will be applied.

During the last decade, there were considerable publications on the stability criteria for systems in the form of $\dot{x}(t) = Ax(t) + Bx(t - t_{\mathrm{d}})$, refer to [1,15,19] and the references therein. Based on these results, robust controller design for uncertain linear systems with time-delay state terms were discussed in [2,20]. For linear systems with delay in control, a famous deadtime compensation method is the Smith predictor [17], which converts the design problem for a process with delay to one without delay by eliminating time delay from the characteristic equation of the closed-loop system. However, the Smith predictor is for open-loop stable systems and cannot handle unmeasurable disturbances. Fuller [4] investigated an optimal regulator problem for

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processes with delay in control. More recently in [9], Smith predictor control was combined with input–output linearization to design a controller for nonlinear time-delay systems.

For systems with the presence of persistent disturbances, H_∞ control has been widely used as it provides explicit performance index in the sense of L_2 gain. With the development of the backstepping technique [10], H_∞ control for lower-triangular systems has been intensively discussed recently, see [13,6,7]. It was shown that by exploring the structural information of the systems, the complicated controller design problem could be simplified by a recursive design procedure.

H_∞ control for systems with delay in input has been discussed in [16,8,9]. In [8], robust stabilization results were given for linear systems with delay in control against additive perturbations, where the H_∞ control problem for a delay system was converted to a problem for a delay-free system. However, the application of the method is limited since the equivalent mapping from the input to the output is only valid under some underlying conditions of the augmented systems. Another interesting work in robust H_∞ control of time-delay systems was reported recently in [3]. State-delayed systems were considered, a delay-dependent Lyapunov function was chosen and the linear matrix inequality (LMI) method was used to construct the control law.

In this paper, we consider the H_∞ control problem for lower-triangular structural systems with delay in control. Backstepping is used as a design tool for constructing the energy function which is nonlinear and delay-dependent. A recursive design procedure is presented and the final control law is dependent on the delay constant t_d. The closed-loop system maintains an L_2 gain from the disturbance input to the output with internal stability. Naturally, the performance index, the L_2 gain, is delay-dependent as well. Advanced dissipativity control theory serves as the underlying theory. It is the first to apply backstepping technique with time-delay characteristics.

The rest of the paper is organized as follows: In Section 2, we give the system configuration and define the main problem concerned in the paper. Then in Section 3, a recursive controller design procedure is presented using backstepping and our main theorem is given. Section 4 presents an example to illustrate the control performance. Finally the paper is concluded in Section 5.

Notations: The notation used in this paper is standard. $|\cdot|$ denotes the usual Euclidean norm for vectors. We say that $z:(0,T)\to\Re^k$ is in $L_2(0,T)$ if $\int_0^T |z(t)|^2\,\mathrm{d}t<\infty$. $\lambda_{\max}(P)$ and $\lambda_{\min}(P)$ denote the maximum and the minimum eigenvalue of any square matrix P.

2. Problem statement

We consider a class of systems in the following state-space form:

$$\begin{aligned}
\dot z &= Az + B(\xi_1 + p_0\omega),\\
\dot\xi_1 &= \xi_2 + p_1\omega,\\
\dot\xi_2 &= \xi_3 + p_2\omega,\\
&\vdots\\
\dot\xi_n &= u(t-t_d) + p_n\omega,\\
y &= z_1,
\end{aligned}\tag{1}$$

where

$$A=\begin{bmatrix} 0 & 1 & & 0\\ & & \ddots & \\ 0 & 0 & & 1\\ 0 & 0 & \dots & 0\end{bmatrix},\qquad B=\begin{bmatrix}0\\ \vdots\\ 0\\ 1\end{bmatrix},$$

$z=(z_1,\dots,z_\kappa)\in\Re^\kappa$, $\xi=(\xi_1,\dots,\xi_n)\in\Re^n$, $[z,\xi]^T$ is the state vector, $u\in\Re$ is the control input, $\omega\in\Re^m$ is the disturbance input, $y\in\Re$ is the to-be-controlled output; p_i $(i=0,1,\dots,n)$ is a constant matrix with appropriate dimension, and t_d is the time-delay constant.

The problem discussed in this paper is defined as follows.

H_∞ **Control Problem.** Find a feedback control law $u(t-t_d)=u(z(t-t_d),\xi(t-t_d))$ for system (1), such that the closed-loop system is globally uniformly asymptotically stable (GUAS) when $\omega=0$. And with the nonzero disturbance $\omega\in L_2(0,T)$, $\forall T\geqslant 0$, the following dissipation inequality is satisfied:

$$\int_0^T |y|^2\,dt \leqslant \mu\int_0^T |w|^2\,dt + v(z(\bar t_0),\xi(\bar t_0),\bar t_0), \tag{2}$$

where μ is a positive constant, v is a positive semidefinite function and $z(\bar t_0),\xi(\bar t_0)$, are the initial conditions where $\bar t_0\in[-t_d,0]$.

Remark 1. In the dissipation inequality (2), the effect of initial conditions is considered. Since $z(\bar t_0)$ and $\xi(\bar t_0)$ happened in history which cannot be compensated, it is reasonable to be included as initial conditions. This also explains the difference between the dissipation inequality for delayed systems and that for delay-free systems.

3. Main result

In this section, we firstly present a recursive controller design procedure by backstepping. Then we show that with the constructed controller, the H_∞ Control Problem is solved.

Step 0: We start by considering the z-subsystem with ξ_1 as the virtual control input. Choose the storage function

$$V_0(z)=z^T Pz, \tag{3}$$

where P is a positive definite symmetric matrix solving the algebraic Riccati equation:

$$A^T P+PA-2\varepsilon PBB^T P+Q=0, \tag{4}$$

where ε is a positive constant and Q is a positive definite symmetric matrix.

Differentiating (3) along the solution of z-subsystem, we have

$$\begin{aligned}\dot V_0(z) &= 2z^T PAz+2z^T PB\xi_1+2z^T PBp_0\omega\\ &\leqslant 2z^T PAz+2z^T PB\xi_1+\frac{1}{\tau_0}|z^T PBp_0|^2+\tau_0|\omega|^2,\end{aligned} \tag{5}$$

where the inequality $2ab\leqslant a^2+b^2$ $(a,b\in\Re)$ is used in the second inequality, and τ_0 is a positive design parameter.

If we let the virtual control $\xi_1=\xi_1^*(z)$ as

$$\xi_1^*(z)=-B^T Pz-\frac{1}{2\tau_0}p_0p_0^T B^T Pz\triangleq\delta_{10}z \tag{6}$$

then (5) turns to

$$\dot V_0(z)\leqslant -z^T Qz+\tau_0|\omega|^2. \tag{7}$$

Step 1: Augment the z-subsystem with the ξ_1-subsystem, and choose a storage function as

$$V_1(z,\xi_1) = V_0(z) + (\xi_1 - \xi_1^*)^2. \tag{8}$$

Denote

$$\begin{aligned}\dot{\xi}_1^*(z) &= \delta_{10}\dot{z} = \delta_{10}Az + \delta_{10}B\xi_1 + \delta_{10}Bp_0\omega \\ &\stackrel{\triangle}{=} \vartheta_{10}z + \vartheta_{11}\xi_1 + \varrho_1\omega.\end{aligned} \tag{9}$$

Differentiating V_1 along the solutions of the (z,ξ_1)-subsystem yields

$$\begin{aligned}\dot{V}_1(z,\xi_1) &\leqslant -z^{\mathrm{T}}Qz + \tau_0|\omega|^2 + 2z^{\mathrm{T}}PB(\xi_1 - \xi_1^*) + 2(\xi_1 - \xi_1^*)(\xi_2 + p_1\omega - \dot{\xi}_1^*) \\ &= -z^{\mathrm{T}}Qz + \tau_0|\omega|^2 + 2(\xi_1 - \xi_1^*)(\xi_2 + z^{\mathrm{T}}PB - \vartheta_{10}z - \vartheta_{11}\xi_1 - \varrho_1\omega + p_1\omega) \\ &\leqslant -z^{\mathrm{T}}Qz + \tau_0|\omega|^2 + 2(\xi_1 - \xi_1^*)(\xi_2 + z^{\mathrm{T}}PB - \vartheta_{10}z - \vartheta_{11}\xi_1) \\ &\quad + \frac{1}{\tau_1}|\xi_1 - \xi_1^*|^2|p_1 - \varrho_1|^2 + \tau_1|\omega|^2,\end{aligned} \tag{10}$$

where τ_1 is a positive constant.

Choose the virtual control $\xi_2 = \xi_2^*$ as

$$\begin{aligned}\xi_2^*(z,\xi_1) &= -B^{\mathrm{T}}Pz - c_1(\xi_1 - \xi_1^*) + \vartheta_{10}z + \vartheta_{11}\xi_1 - \frac{1}{\tau_1}(\xi_1 - \xi_1^*)|p_1 - \varrho_1|^2 \\ &\triangleq \delta_{20}z + \delta_{21}\xi_1,\end{aligned} \tag{11}$$

where $c_1 > 0$.

Denote $\tilde{\xi}_1 = \xi_1 - \xi_1^*$, then (10) turns to

$$\dot{V}_1(z,\xi_1) \leqslant -z^{\mathrm{T}}Qz - 2c_1\tilde{\xi}_1^2 + (\tau_0 + \tau_1)|\omega|^2. \tag{12}$$

Claim. *Given any index $2 \leqslant l \leqslant n-1$, for the system*

$$\begin{aligned}\dot{z} &= Az + B(\xi_1 + p_0\omega), \\ \dot{\xi}_1 &= \xi_2 + p_1\omega, \\ \dot{\xi}_2 &= \xi_3 + p_2\omega, \\ &\vdots\end{aligned} \tag{13}$$

$$\dot{\xi}_l = \xi_{l+1} + p_{l+1}\omega, \tag{14}$$

there exist $l+1$ smooth functions

$$\xi_i^* = \xi_i^*(z,\xi_1,\ldots,\xi_{i-1}) = \delta_{i0}z + \delta_{i1}\xi_1 + \cdots + \delta_{i,i-1}\xi_{i-1}, \quad 1 \leqslant i \leqslant l+1$$

such that in new coordinates

$$\tilde{\xi}_i = \xi_i - \xi_i^*(z,\xi_1,\ldots,\xi_{i-1}), \quad 1 \leqslant i \leqslant l$$

the storage function

$$V_l = V_0 + \sum_{i=1}^{l} \tilde{\xi}_i^2 \tag{15}$$

has time derivative, *with* $\xi_{l+1} = \xi_{l+1}^*$, *satisfying the dissipation inequality*

$$\dot{V}_l(z, \xi_1) \leqslant -z^{\mathrm{T}} Q z - \sum_{i=1}^{l} 2c_i \tilde{\xi}_i^2 + \sum_{i=1}^{l} \tau_i |\omega|^2. \tag{16}$$

The proof of the Claim is straightforward and is given in the appendix.

Step n: Let

$$V_n(z, \xi_1, \ldots, \xi_n) = V_{n-1} + (\xi_n - \xi_n^*)^2 + V_{\mathrm{d}}(t, z, \xi), \tag{17}$$

where

$$\begin{aligned} V_{\mathrm{d}}(t, z, \xi) = {} & \int_{-t_{\mathrm{d}}}^{0} \int_{t+\theta}^{t} \left[r_0 z^{\mathrm{T}}(s) z(s) + \sum_{i=1}^{n} r_i \xi_i^{\mathrm{T}}(s) \xi_i(s) + \kappa \omega(s)^{\mathrm{T}} \omega(s) \right] \mathrm{d}s \, \mathrm{d}\theta \\ & + \int_{-t_{\mathrm{d}}}^{0} \int_{t-t_{\mathrm{d}}+\theta}^{t} \left[l_0 z^{\mathrm{T}}(s) z(s) + \sum_{i=1}^{n} l_i \xi_i^{\mathrm{T}}(s) \xi_i(s) \right] \mathrm{d}s \, \mathrm{d}\theta \end{aligned} \tag{18}$$

and $r_0, r_i, l_0, l_i, \kappa$ are positive design parameters.

From Step $n-1$ by the Claim, $\xi_n^* = \delta_{n0} z + \delta_{n1} \xi_1 + \cdots + \delta_{n,n-1} \xi_{n-1}$. Denote

$$\begin{aligned} \dot{\xi}_n^*(z, \xi_1, \ldots, \xi_{n-1}) &= \left(\frac{\partial \xi_n^*}{\partial z} \dot{z} + \frac{\partial \xi_n^*}{\partial \xi_1} \dot{\xi}_1 + \cdots + \frac{\partial \xi_n^*}{\partial \xi_{n-1}} \dot{\xi}_{n-1} \right) \\ &\triangleq \vartheta_{n0} z + \vartheta_{n1} \xi_1 + \cdots + \vartheta_{nn} \xi_n + \varrho_n \omega. \end{aligned} \tag{19}$$

Differentiating V_n along the solutions of system (1), we have

$$\begin{aligned} \dot{V}_n &\leqslant -z^{\mathrm{T}} Q z - \sum_{i=1}^{n-1} 2c_i \tilde{\xi}_i^2 + \sum_{i=1}^{n-1} \tau_i |\omega|^2 + 2\tilde{\xi}_n [u(t - t_{\mathrm{d}}) + p_n \omega + \tilde{\xi}_{n-1} - \dot{\xi}_n^*] + \dot{V}_{\mathrm{d}}(t, z, \xi) \\ &= -z^{\mathrm{T}} Q z - \sum_{i=1}^{n-1} 2c_i \tilde{\xi}_i^2 + \sum_{i=1}^{n-1} \tau_i |\omega|^2 + 2\tilde{\xi}_n [u(t - t_{\mathrm{d}}) + L_0 z + \sum_{i=1}^{n} L_i \xi_i] + 2\tilde{\xi}_n (p_n - \varrho_n) \omega \\ &\quad + \dot{V}_{\mathrm{d}}(t, z, \xi), \end{aligned} \tag{20}$$

where $\tilde{\xi}_n = \xi_n - \xi_n^*$, and

$$\begin{aligned} L_0 &= -\delta_{n-1,0} - \vartheta_{n0}, \\ L_i &= \begin{cases} -\delta_{n-1,i} - \vartheta_{ni}, & 1 \leqslant i \leqslant n-2, \\ 1 - \vartheta_{ni}, & i = n-1, \\ -\vartheta_{ni}, & i = n. \end{cases} \end{aligned} \tag{21}$$

Note that

$$2\tilde{\xi}_n (p_n - \varrho_n) \omega \leqslant \frac{1}{\tau_n} \tilde{\xi}_n^2 |p_n - \varrho_n|^2 + \tau_n |\omega|^2. \tag{22}$$

Denote

$$u^*(z,\xi) = c_n\tilde{\xi}_n + L_0 z + \sum_{i=1}^{n} L_i\xi_i + \frac{1}{2\tau_n}\tilde{\xi}_n|p_n - \varrho_n|^2$$
$$\triangleq K_0 z + K_1\xi_1 + K_2\xi_2 + \cdots + K_n\xi_n \tag{23}$$

then (20) turns to

$$\dot{V}_n \leqslant -z^{\mathrm{T}}Qz - \sum_{i=1}^{n} 2c_i\tilde{\xi}_i^2 + \sum_{i=1}^{n}\tau_i|\omega|^2 + 2\tilde{\xi}_n[u(t-t_{\mathrm{d}}) + u^*(z,\xi)] + \dot{V}_{\mathrm{d}}(t,z,\xi). \tag{24}$$

Since $z(t), \xi_i(t),\ 1 \leqslant i \leqslant n$ are continuously differentiable for $t \geqslant 0$, we have

$$z(t-t_{\mathrm{d}}) = z(t) - \int_{-t_{\mathrm{d}}}^{0} \dot{z}(t+\theta)\,\mathrm{d}\theta$$
$$= z(t) - \int_{-t_{\mathrm{d}}}^{0} [Az(t+\theta) + B(\xi_1(t+\theta) + p_0\omega(t+\theta))]\,\mathrm{d}\theta, \tag{25}$$

$$\xi_i(t-t_{\mathrm{d}}) = \xi_i(t) - \int_{-t_{\mathrm{d}}}^{0} [\xi_{i+1}(t+\theta) + p_i\omega(t+\theta)]\,\mathrm{d}\theta, \quad (1 \leqslant i \leqslant n-1), \tag{26}$$

$$\xi_n(t-t_{\mathrm{d}}) = \xi_n(t) - \int_{-t_{\mathrm{d}}}^{0} [u(t-t_{\mathrm{d}}+\theta) + p_n\omega(t+\theta)]\,\mathrm{d}\theta. \tag{27}$$

Note that by using (25)–(27), the initial condition of system (1) shifts from $[-t_{\mathrm{d}}, 0]$ to $[-2t_{\mathrm{d}}, 0]$, while the system stability remains same [5, p. 131].

Now choose the true delayed control law

$$u(t-t_{\mathrm{d}}) = -K_0 z(t-t_{\mathrm{d}}) - K_1\xi_1(t-t_{\mathrm{d}}) - K_2\xi_2(t-t_{\mathrm{d}}) - \cdots - K_n\xi_n(t-t_{\mathrm{d}}). \tag{28}$$

Substitute (25)–(27), (28) into (24),

$$\dot{V}_n \leqslant -z^{\mathrm{T}}Qz - \sum_{i=1}^{n} 2c_i\tilde{\xi}_i^2 + \sum_{i=1}^{n}\tau_i|\omega|^2 + \eta_1(t,z,\xi) + \eta_2(t,z,\xi) + \eta_3(t,z,\xi) + \dot{V}_{\mathrm{d}}(t,z,\xi), \tag{29}$$

where

$$\eta_1(t,z,\xi) = 2\tilde{\xi}_n \int_{-t_{\mathrm{d}}}^{0} \left[K_0Az(t+\theta) + K_0B\xi_1(t+\theta) + \sum_{i=1}^{n-1} K_i\xi_{i+1}(t+\theta)\right]\mathrm{d}\theta,$$

$$\eta_2(t,z,\xi) = 2\tilde{\xi}_n \int_{-t_{\mathrm{d}}}^{0} \left[K_nK_0z(t-t_{\mathrm{d}}+\theta) + \sum_{i=1}^{n} K_nK_i\xi_i(t-t_{\mathrm{d}}+\theta)\right]\mathrm{d}\theta,$$

$$\eta_3(t,z,\xi) = 2\tilde{\xi}_n \int_{-t_{\mathrm{d}}}^{0} (K_0p_0 + \sum_{i=1}^{n} K_ip_i)\omega(t+\theta)\,\mathrm{d}\theta. \tag{30}$$

Using $2ab \leqslant a^2 + b^2 (a, b \in \mathfrak{R})$ again, we obtain

$$\eta_1(t,z,\xi) \leqslant t_{\mathrm{d}}\left(\frac{1}{r_0}|K_0A|^2 + \frac{1}{r_1}|K_0B|^2 + \sum_{i=1}^{n-1}\frac{1}{r_{i+1}}|K_i|^2\right)\tilde{\xi}_n^2$$

$$+\int_{-t_d}^{0}\left[r_0|z(t+\theta)|^2+\sum_{i=1}^{n} r_i|\xi_i(t+\theta)|^2\right]\mathrm{d}\theta,$$

$$\eta_2(t,z,\xi)\leqslant t_d\left(\frac{1}{l_0}|K_nK_0|^2+\sum_{i=1}^{n}\frac{1}{l_i}|K_nK_i|^2\right)\tilde{\xi}_n^2$$

$$+\int_{-t_d}^{0}\left[l_0|z(t-t_d+\theta)|^2+\sum_{i=1}^{n} l_i|\xi_i(t-t_d+\theta)|^2\right]\mathrm{d}\theta,$$

$$\eta_3(t,z,\xi)\leqslant t_d\frac{1}{\kappa}(K_0p_0+\sum_{i=1}^{n}K_ip_i)\tilde{\xi}_n^2+\int_{-t_d}^{0}\tau|\omega(t+\theta)|^2\,\mathrm{d}\theta. \tag{31}$$

Notice that

$$\dot V_d(t,z,\xi)=t_dr_0z^{\mathrm T}z+\sum_{i=1}^{n}t_dr_i\xi_i^{\mathrm T}\xi_i+t_d\kappa\omega^{\mathrm T}\omega$$

$$-\int_{-t_d}^{0}\left[r_0z^{\mathrm T}(t+\theta)z(t+\theta)+\sum_{i=1}^{n}r_i\xi_i^{\mathrm T}(t+\theta)\xi_i(t+\theta)+\tau\omega(t+\theta)^{\mathrm T}\omega(t+\theta)\right]\mathrm{d}\theta$$

$$+t_dl_0z^{\mathrm T}z+\sum_{i=1}^{n}t_dl_i\xi_i^{\mathrm T}\xi_i$$

$$-\int_{-t_d}^{0}\left[l_0z^{\mathrm T}(t-t_d+\theta)z(t-t_d+\theta)+\sum_{i=1}^{n}l_i\xi_i^{\mathrm T}(t-t_d+\theta)\xi_i(t-t_d+\theta)\right]\mathrm{d}\theta. \tag{32}$$

Substitute (31), (32) into (29),

$$\dot V_n\leqslant -z^{\mathrm T}Qz-\sum_{i=1}^{n}2c_i\tilde{\xi}_i^2+\sum_{i=1}^{n}\tau_i|\omega|^2$$

$$+t_d(r_0+l_0)|z|^2+\sum_{i=1}^{n}t_d(r_i+l_i)|\xi_i|^2+t_d\phi\tilde{\xi}_n^2+t_d\kappa|\omega|^2, \tag{33}$$

where

$$\phi=\left(\frac{1}{r_0}|K_0A|^2+\frac{1}{r_1}|K_0B|^2+\sum_{i=1}^{n-1}\frac{1}{r_{i+1}}|K_i|^2\right)$$

$$+\left(\frac{1}{l_0}|K_nK_0|^2+\sum_{i=1}^{n}\frac{1}{l_i}|K_nK_i|^2\right)+\frac{1}{\kappa}(K_0p_0+\sum_{i=1}^{n}K_ip_i). \tag{34}$$

Since the mapping

$$\Theta:(z,\xi_1,\xi_2,\ldots,\xi_n)\mapsto(z,\tilde{\xi}_1,\tilde{\xi}_2,\ldots,\tilde{\xi}_n)$$

is a linear transformation whose Jacobian matrix is lower triangular with all diagonal components equal to the constant one, it is easy to get

$$\xi_i=\tilde{\xi}_i+\varphi_{i-1}\tilde{\xi}_{i-1}+\cdots+\varphi_1\tilde{\xi}_1+\varphi_0z, \tag{35}$$

where φ_0 is a constant matrix, and $\varphi_1,\ldots,\varphi_{i-1}$ are constants.

Applying

$$(a_1 + a_2 + \cdots + a_k)^2 \leqslant k(a_1^2 + a_2^2 + \cdots + a_k^2), \quad a_1, a_2, \ldots, a_k \in \mathfrak{R}$$

to (35),

$$|\xi_i|^2 \leqslant (i+1)(|\varphi_0|^2|z|^2 + \varphi_1^2\tilde{\xi}_1^2 + \cdots + \varphi_{i-1}^2\tilde{\xi}_{i-1}^2 + \tilde{\xi}_i^2). \tag{36}$$

Therefore (33) turns to

$$\dot{V}_n \leqslant -\alpha|z|^2 - \sum_{i=1}^{n} \beta_i \tilde{\xi}_i^2 + \tau|\omega|^2, \tag{37}$$

where

$$\alpha = \lambda_{\min}(Q) - t_{\mathrm{d}}\left[(r_0 + l_0) + \sum_{i=1}^{n}(r_i + l_i)(i+1)|\varphi_0|^2\right],$$

$$\beta_i = \begin{cases} 2c_i - t_{\mathrm{d}}(r_i + l_i)(i+1)\varphi_i^2, & 1 \leqslant i \leqslant n-1, \\ 2c_i - t_{\mathrm{d}}[(r_i + l_i)(i+1) + \phi], & i = n, \end{cases}$$

$$\tau = \sum_{i=1}^{n} \tau_i + t_{\mathrm{d}}\kappa. \tag{38}$$

It can be seen that α, β_i $(1 \leqslant i \leqslant n), \tau$ are dependent on t_{d}. For certain small t_{d}, it is possible to find positive design parameters $\varepsilon, Q, c_i, r_0, r_i, l_0, l_i, \tau_i, \kappa$ to ensure that $\alpha, \beta_i > 0$.

Now we are in the position to state our main result.

Theorem 1. *The feedback control law* (28) *solves the H_∞ Control Problem if there exist positive constants/matrix $\varepsilon, Q, c_i, r_0, r_i, l_0, l_i, \tau_i, \kappa$, $(1 \leqslant i \leqslant n)$ such that α, β_i are positive as defined in* (38).

The proof of the theorem is given in the appendix.

Remark 2. The controller design method proposed above is a recursive procedure. It follows the backstepping design where energy function and virtual control law are stepwisely constructed. The delay in the input makes the energy function more complicated than that for delay-free systems. It includes an integral term of the states within the delayed time period. The final dissipation inequality (37) shows the energy dissipating along the closed-loop trajectory. The design is delay-dependent.

Remark 3. The integral form of energy function (18) was firstly proposed in [12] and recently used in [3]. While in [3] state-delayed systems were discussed and initial conditions were assumed to be zero, the present paper deals with systems with delay in control and initial conditions are explicitly addressed in the dissipation inequality (2). It is also noted that the proving method in [3] is only valid for the L_2 space defined on $[0, \infty)$ instead of that on $[0, T]$, $\forall T > 0$.

4. An illustrated example

We consider the following lower-triangular system:

$$\begin{aligned} \dot{z}_1 &= z_2, \\ \dot{z}_2 &= \xi + 0.2\omega, \\ \dot{\xi} &= u(t - t_{\mathrm{d}}) + 0.5\omega. \end{aligned} \tag{39}$$

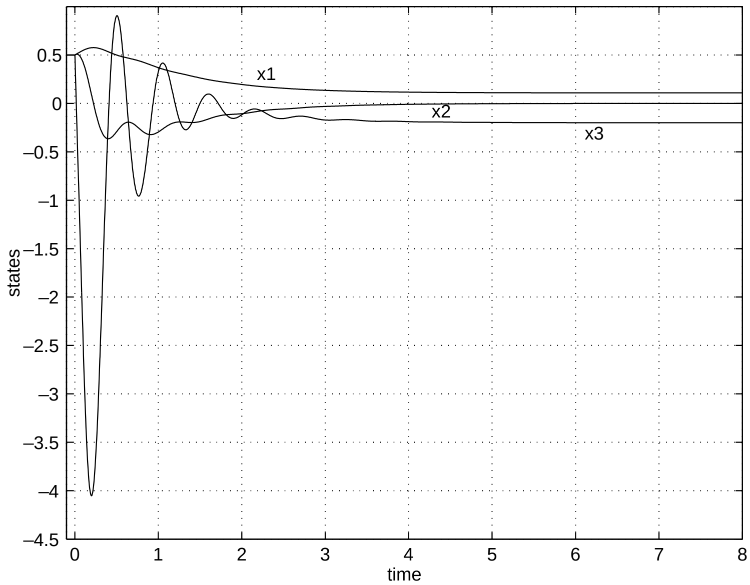

Fig. 1. Closed-loop system response by the proposed H_∞ control: $[x1; x2; x3] = [z_1; z_2; \xi]$.

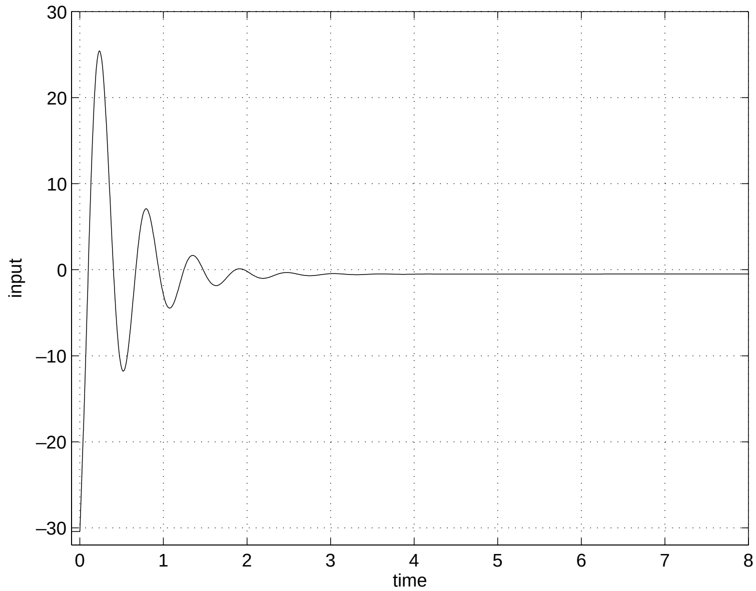

Fig. 2. Control input response.

Following the design procedure described in Section 3, and choosing the design parameters

$$\varepsilon = 20, \quad Q = I, \quad c_1 = 4, \quad \tau_0 = \tau_1 = 0.5,$$

we obtain the control law

$$u(t - t_{\mathrm{d}}) = -22.3741 z_1(t - t_{\mathrm{d}}) - 23.8378 z_2(t - t_{\mathrm{d}}) - 9.6352 \xi(t - t_{\mathrm{d}}).$$

Simulation results of the closed-loop system is shown in Figs. 1 and 2, where $t_{\mathrm{d}} = 0.1$, $\omega = 1$, and the initial condition is $[z_1\ z_2\ \xi]^{\mathrm{T}} = [0.5\ 0.5\ 0.5]^{\mathrm{T}}$. Satisfactory performance for all states can be observed.

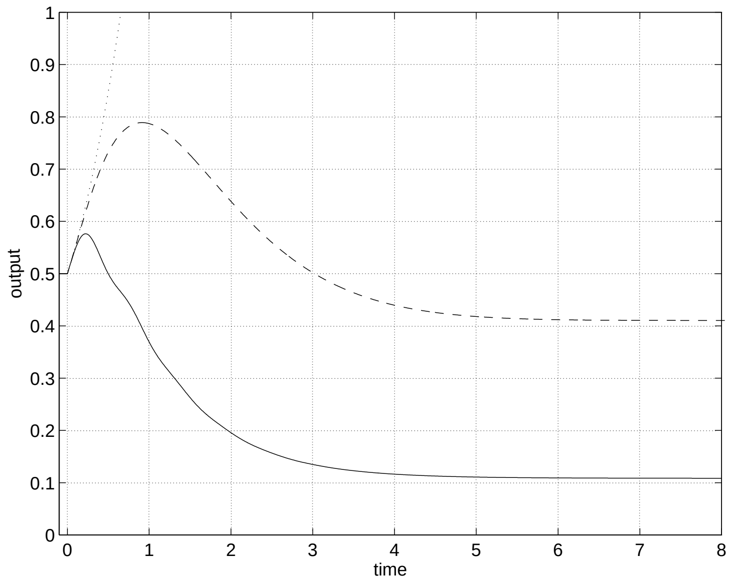

Fig. 3. Comparison of output responses: solid line — proposed H_∞ control; dashed line — LQ control; dotted line — open-loop system.

To compare our controller to the open-loop system and the standard LQ design method, we show in Fig. 3 the output trajectories under the three situations. It can be seen that the open-loop system is unstable; the LQ design method stabilizes the system but cannot attenuate the disturbance's effect (as it was not considered in the design); and the H_∞ control design by the proposed method ensures the stability and achieves disturbance attenuation to the output.

5. Conclusions

The H_∞ control problem for the lower-triangular systems with delay in control has been studied. An explicit feedback control law has been built, and the closed-loop system maintains the L_2-gain from the disturbance input to the output with internal stability. Backstepping technique was applied to the structural system. The design is delay-dependent. Simulation results on a third-order example system have demonstrated its good performance compared to the open-loop system and the standard LQ design method.

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Appendix

Proof of the Claim. As stated previously, the Claim holds for system $l=1$. Assume that the Claim is true for system $l=k-1$, we wish to prove the claim for system $l=k$.

Consider the storage function

$$V_k = V_{k-1} + \tilde{\xi}_k^2. \tag{A.1}$$

Differentiating it along the solutions of the $(z, \xi_1, \ldots, \xi_k)$-subsystem in (14), we have

$$\dot{V}_k \leqslant -z^{\mathrm{T}}Qz - \sum_{i=1}^{k-1} 2c_i \tilde{\xi}_i^2 + \sum_{i=1}^{k-1} \tau_i |\omega|^2 + 2\tilde{\xi}_k[\tilde{\xi}_{k-1} + \xi_{k+1} + p_k\omega - \dot{\xi}_k^*], \tag{A.2}$$

where $\tilde{\xi}_k = \xi_k - \xi_k^*$.

Note that

$$\begin{aligned}\dot{\xi}_k^* &= \delta_{k0}\dot{z} + \delta_{k1}\dot{\xi}_1 + \cdots + \delta_{k,k-1}\dot{\xi}_{k-1} \\ &\triangleq \vartheta_{k0}z + \vartheta_{k1}\xi_1 + \cdots + \vartheta_{kk}\xi_k + \varrho_k\omega.\end{aligned} \tag{A.3}$$

Substituting (A.3) into (A.2), and collecting similar terms, we get

$$\dot{V}_k \leqslant -z^{\mathrm{T}}Qz - \sum_{i=1}^{k-1} 2c_i \tilde{\xi}_i^2 + \sum_{i=1}^{k-1} \tau_i |\omega|^2 + 2\tilde{\xi}_k \left[\xi_{k+1} + \sigma_0 z + \sum_{i=1}^{k} \sigma_i \xi_i\right] + 2\tilde{\xi}_k(p_k - \varrho_k)\omega, \tag{A.4}$$

where

$$\begin{aligned}\sigma_0 &= -\delta_{k-1,0} - \vartheta_{k0}, \\ \sigma_i &= \begin{cases} -\delta_{k-1,i} - \vartheta_{ki}, & 1 \leqslant i \leqslant k-2, \\ 1 - \vartheta_{ki}, & i = k-1, \\ -\vartheta_{ki}, & i = k. \end{cases}\end{aligned} \tag{A.5}$$

Note that the last term of (A.4) turns to

$$2\tilde{\xi}_k(p_k - \varrho_k)\omega \leqslant \frac{1}{\tau_k}\tilde{\xi}_k^2 |p_k + \varrho_k|^2 + \tau_k |\omega|^2, \tag{A.6}$$

where τ_k is a positive design parameter.

Choose the virtual control $\xi_{k+1} = \xi_{k+1}^*$ as

$$\begin{aligned}\xi_{k+1}^* &= -c_k\tilde{\xi}_k - \sigma_0 z - \sum_{i=1}^{k} \sigma_i \xi_i - \frac{1}{2\tau_k}\tilde{\xi}_k |p_k + \varrho_k|^2 \\ &\triangleq \delta_{k+1,0}z + \delta_{k+1,1}\xi_1 + \cdots + \delta_{k+1,k}\xi_k,\end{aligned} \tag{A.7}$$

where c_k is a positive design parameter, then we get

$$\dot{V}_k \leqslant -z^{\mathrm{T}}Qz - \sum_{i=1}^{k} 2c_i \tilde{\xi}_i^2 + \sum_{i=1}^{k} \tau_i |\omega|^2. \tag{A.8}$$

This completes the proof of the Claim. □

Proof of Theorem 1. From the design procedure described in Section 3, we have the following Lyapunov function,

$$\begin{aligned}V_n(t, z, \xi) &= z^{\mathrm{T}}Pz + \sum_{i=1}^{n} \tilde{\xi}_i^2 \\ &\quad + \int_{-t_{\mathrm{d}}}^{0} \int_{t+\theta}^{t} \left[r_0 z^{\mathrm{T}}(s)z(s) + \sum_{i=1}^{n} r_i \xi_i^{\mathrm{T}}(s)\xi_i(s) + \kappa\omega(s)^{\mathrm{T}}\omega(s)\right] \mathrm{d}s\,\mathrm{d}\theta \\ &\quad + \int_{-t_{\mathrm{d}}}^{0} \int_{t-t_{\mathrm{d}}+\theta}^{t} \left[l_0 z^{\mathrm{T}}(s)z(s) + \sum_{i=1}^{n} l_i \xi_i^{\mathrm{T}}(s)\xi_i(s)\right] \mathrm{d}s\,\mathrm{d}\theta.\end{aligned} \tag{A.9}$$

Due to (35), V_n is positive definite in the coordinates $(z, \tilde{\xi})$. Its derivative along the closed-loop trajectory is

$$\dot{V} \leqslant -\alpha|z|^2 - \sum_{i=1}^{n} \beta_i \tilde{\xi}_i^2 + \tau|\omega|^2. \tag{A.10}$$

If α, β_i, τ are all positive, when $\omega = 0$, $\dot{V}$ is negative definite. So the transferred states $(z, \tilde{\xi})$ is GUAS. Since (35), the original coordinates of system (1) is GUAS too.

When $\omega \neq 0$, taking the integral along time t for both sides of (A.10) and simplifying, we obtain

$$\int_0^T \alpha|z|^2 \,\mathrm{d}t \leqslant \tau \int_0^T |\omega|^2 \,\mathrm{d}t + V_n(0, z(0), \xi(0)). \tag{A.11}$$

Therefore,

$$\int_0^T |y|^2 \,\mathrm{d}t \leqslant \mu \int_0^T |w|^2 \,\mathrm{d}t + v(z(\bar{t}_0), \xi(\bar{t}_0), \bar{t}_0), \tag{A.12}$$

where

$$\mu = \tau/\alpha,$$
$$v(z(\bar{t}_0), \xi(\bar{t}_0), \bar{t}_0) = V_\mathrm{d}(0, z(0), \xi(0))/\alpha. \tag{A.13}$$

This completes the proof of the theorem. □

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