Ma 221 - Exam II review

Second Order Differential Equations

Form of general solution

\[ y_h = c_1 y_1 + c_2 y_2 \]

where \( y_1 \) and \( y_2 \) are linearly independent solutions of the homogeneous equation and

\[ y = y_h + y_p \]

where \( y_p \) is a [particular] solution of the non-homogeneous equation.

Wronskian - provides a test for linear independence of solutions

\[ W[y_1, y_2] = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}. \]

\[ W[y_1, y_2] \neq 0 \] for linearly independent solutions of the homogeneous d.e.

Homogeneous D.E.

Constant coefficients -

\[ ay'' + by' + cy = 0 \]

Solve auxiliary (characteristic) equation -

\[ p(r) = ar^2 + br + c = 0 \]

2 real roots

\[ r = r_1, r_2 \]

\[ y_h = c_1 e^{r_1 x} + c_2 e^{r_2 x} \]

repeated real roots

\[ r = r_1 \]

\[ y_h = (c_1 + c_2 x)e^{r_1 x} \]

2 complex roots

\[ r = a \pm i\beta \]

\[ y_h = (c_1 \cos \beta x + c_2 \sin \beta x)e^{ax} \]
Cauchy-Euler D.E. -

\[ ax^2y'' + bxy' + cy = 0 \]

Solve auxiliary (indicial) equation -

\[ am^2 + (b - a)m + c = 0 \]

Or one can divide by \( a \) and put the equation in the form

\[ x^2y'' + pxy' + qy = 0 \]

Then the auxiliary (indicial) equation that one must solve for \( m \) is

\[ m^2 + (p - 1)m + q = 0 \]

2 real roots

\[ m = m_1, m_2 \]

\[ y_h = c_1 x^{m_1} + c_2 x^{m_2} \]

repeated real roots

\[ m = m_1 \]

\[ y_h = (c_1 + c_2 \ln x)x^{m_1} \]

2 complex roots

\[ m = \alpha \pm i\beta \]

\[ y_h = (c_1 \cos \beta \ln x + c_2 \sin \beta \ln x)x^\alpha \]
Non-homogeneous D.E.

Undetermined coefficients

Constant coefficient d.e.

\[ ay'' + by' + cy = f(x) \]
\[ f(x) = Ae^{ax} \]
\[ f(x) = (A \cos \beta x + B \sin \beta x)e^{ax} \]
\[ f(x) = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n \]

and products of above

Be careful if \( f(x) \) is a solution of the homogeneous equation.

In the case with \( f(x) = Ae^{ax} \), the following formulae give the solution.

\[
y(x) = \begin{cases} 
\frac{A}{p(a)} e^{ax} & p(a) \neq 0 \\
\frac{A}{p'(a)} xe^{ax} & p(a) = 0, p'(a) \neq 0 \\
\frac{A}{p''(a)} x^2e^{ax} & p(a) = p'(a) = 0
\end{cases}
\]

Cauchy-Euler D.E.

\[ ax^2y'' + bxy' + cy = f(x) \]
\[ f(x) = Ax^p \]

If \( f(x) \) is a solution of the homogeneous equation, use variation of parameters.

Variation of parameters

\[ a(x)y'' + b(x)y' + c(x)y = f(x) \]
\[ y = v_1y_1 + v_2y_2 \]
\[ y_1y_1' + y_2y_2' = 0 \]
\[ y_1y_1' + y_2y_2' = \frac{f(x)}{a(x)} \]
\[ v_1' = -\frac{f(x)y_2}{a(x)(y_1y_2' - y_2y_1')} \]
\[ v_2' = \frac{f(x)y_1}{a(x)(y_1y_2' - y_2y_1')} \]