Example 1

1. Cls.
   \[ f_k = k \times x_s \quad f_d = b \times v_d \]

2. GC
   \[ V_s + V_d = 0 \]

3. FBD
   \[ \sum F_x = ma \quad -f_s + f_d = 0 \quad (\text{bolt connection}) \]

4. State variables: \( x_s \) (only one spring)

5. \( x_s' = \) ?
   
   \[ x_s' = V_s = -V_d = -\frac{f_d}{b} = -\frac{f_s}{b} = -\frac{k \times x_s}{b} \]

   \[ x_s' = (-\frac{k}{b}) x_s \]

   ← this is correct form
Thus we have determined the state equation describing the behavior of the system over time.

If the system was disturbed, the state equation provides the state of the system as a function of time.

State equation for example 1: \( x_s' = -\frac{1}{b} x_s \).

This equation can be solved analytically or numerically. To find the analytical solution (DEQs):

Assume \( x_s(t) = A e^{rt} \) is a solution.

\( x_s'(t) = r A e^{rt} \)

If this "guess" is correct, then it must satisfy the state equation:

\( x_s' = -\frac{1}{b} x_s \)

\( r A e^{rt} = -\frac{1}{b} (A e^{rt}) \)

\( r = -\frac{1}{b} \) (← this must be true for our guess to work)
Thus our solution to the state equation is

\[ x_s(t) = A e^{-\frac{K_R t}{b}} \] — general solution

Here 'A' is a constant which can only be determined from the initial conditions (i.e. the perturbation). There is one unknown; thus we need one initial condition.

In math terminology... we have solved for the general solution of the DEQ. If we have the initial conditions, we can solve for the particular solution.

Assume that the initial elongation of the spring is \( x_s(t=0) = 2 \). What is the particular solution?

\[ x_s(t) = A e^{-\frac{K_R t}{b}} \]

If at \( t=0 \), \( x_s = 2 \), then

\[ 2 = A e^{-\frac{K_R (0)}{b}} = A (1) \]

\[ A = 2 \]

\[ x_s(t) = 2 e^{-\frac{K_R t}{b}} \] — particular solution
Example 2 (Parachute problem)

\[ f_d = b v_d \]

(air resistance)

\[ V_d = V_m \]

\[ \sum F_y = m a_m \]

\[ m g - f_d = m a_m \]

4] \( SV: V_m \)

5] State equation(s):

\[ V_m' = a_m = \frac{1}{m} (m g - f_d) = \frac{1}{m} (m g - b v_d) \]

\[ V_m' = \frac{1}{m} (m g - b v_m) \] is appropriate state equation
EXAMPLE 3

\[ \begin{align*}
X & \rightarrow \quad b \\
& \quad m
\end{align*} \]

1. CL \quad f_d = b v_d

2. GC \quad V_d = V_m

3. \[ \begin{align*}
\frac{t}{g} & \quad \sum F_x = m a \\
- f_d & = m a
\end{align*} \]

4. SV: \quad V_m

5. \[ \begin{align*}
V_m' & = a_m = - \frac{f_d}{m} = - b V_d = - b \frac{V_m}{m}
\end{align*} \]

\[ V_m' = - \frac{b}{m} V_m \]

Analytical solution: Guess \( v_m(t) = Ae^{rt} \)
Follow procedure from before

\[ V_m(t) = A e^{- \frac{b}{m} t} \quad \text{general solution} \]
Example 4

1) CL: \( f_s = k x_s \quad f_d = b v_d \)

2) GC
\( v_s = v_d \)
\( v_s = v_m \)

3) FBD
\[
\begin{align*}
\downarrow \Sigma F &= ma \\
-f_{\text{rope}} + mg &= ma_m
\end{align*}
\]
\[
\begin{align*}
f_{\text{rope}} &= 0 \quad \text{(massless connection)}
\end{align*}
\]

4) SV: \( x_s, v_m \) (Here we have two state variables... need two state equations)

5) Solve for each state equation separately:

\[
x_s' = v_s = v_m \quad \checkmark
\]

\[
v_m' = a_m = \frac{1}{m} \left( -f_{\text{rope}} + mg \right) = \frac{1}{m} \left[ (f_s + f_d) + mg \right]
\]
\[
= \frac{1}{m} \left[ -k x_s - b v_d + mg \right].
\]

\[
v_m' = \frac{1}{m} \left[ -k x_s - b v_m + mg \right] \quad \checkmark
\]
Thus the state equations are:

\[
\begin{align*}
\dot{x}_s &= v_m \\
\dot{v}_m &= \frac{1}{m} (mg - kx_s - bv_m)
\end{align*}
\]

Can write in matrix form as

\[
\begin{pmatrix}
\dot{x}_s \\
\dot{v}_m
\end{pmatrix} = 
\begin{bmatrix}
0 & 1 \\
-k/m & -b/m
\end{bmatrix}
\begin{pmatrix}
x_s \\
v_m
\end{pmatrix} + \begin{pmatrix}
0 \\
q
\end{pmatrix}
\]

Can also write in second order form:

\[
x_s'' = ?
\]

\[
x_s'' = \frac{d}{dt} (x_s') = \frac{d}{dt} (v_m) = v_m'
\]

\[
x_s''' = v_m' = \frac{1}{m} (mg - kx_s - bv_m)
\]

Now \(v_m\) is not a \(5\)th order derivative.

\[
x_s'''' = \frac{1}{m} (mg - kx_s - bv_s')
\]

This is ok; lower order derivative.

Can re-write in standard form:

\[
x_s'' + \frac{b}{m} x_s' + \frac{k}{m} x_s = q
\]

In math terms, this is "non-homogeneous."
EXAMPLE 5: EIGENVALUE PROBLEM EXAMPLE

1) CL: \( f_{s1} = k_1 x_{s1} \quad f_{s2} = k_2 x_{s2} \)

2) GC: \( v_{s1} = v_{m1} \quad v_{s2} = v_{m2} - v_{m1} \)

4) FBD:

\[
\begin{align*}
F_x &= m_1 a_{m1} \\
&= f_{s1} + f_{s2} = m_1 a_{m1}
\end{align*}
\]

\[
\begin{align*}
F_x &= m_2 a_{m2} \\
&= -f_{s2} = m_2 a_{m2}
\end{align*}
\]

5) State variables: \( x_{s1}, x_{s2}, v_{m1}, v_{m2} \) (Four SV's here!)

5) \( x'_{s1} = v_{s1} = v_{m1} \) \( \checkmark \)

\( x'_{s2} = v_{s2} = v_{m2} - v_{m1} \) \( \checkmark \)

\( v'_{m1} = a_{m1} = \frac{1}{m_1} (-f_{s1} + f_{s2}) = \frac{1}{m_1} (-k_1 x_{s1} + k_2 x_{s2}) \)

\( v'_{m2} = a_{m2} = \frac{1}{m_2} (-f_{s2}) = \frac{1}{m_2} (-k_2 x_{s2}) \) \( \checkmark \)
can write in matrix form:

\[
\begin{bmatrix}
X_{s1}' \\
X_{s2}' \\
V_{m1}' \\
V_{m2}'
\end{bmatrix} = \begin{bmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & -1 & 1 \\
-\frac{k_1}{m_1} & \frac{k_2}{m_2} & 0 & 0 \\
0 & -\frac{k_2}{m_2} & 0 & 0
\end{bmatrix} \begin{bmatrix}
X_{s1} \\
X_{s2} \\
V_{m1} \\
V_{m2}
\end{bmatrix}
\]

But more useful to write in second order form:

\[
V_{m1}'' = \frac{d}{dt} (V_{m1}') = \frac{1}{m_1} \left(-k_1 X_{s1}' + k_2 X_{s2}'\right)
= \frac{1}{m_1} \left[-k_1 V_{s1} + k_2 V_{s2}\right]
= \frac{1}{m_1} \left[-k_1 V_{m1} + k_2 (V_{m2} - V_{m1})\right]
\]

\[
V_{m2}'' = \frac{d}{dt} (V_{m2}') = \frac{1}{m_2} \left(-k_2 X_{s2}'\right)
= \frac{1}{m_2} \left(-k_2 V_{s2}\right) = \frac{1}{m_2} \left[-k_2 (V_{m2} - V_{m1})\right]
\]

\[
\begin{bmatrix}
V_{m1}'' \\
V_{m2}''
\end{bmatrix} = \begin{bmatrix}
-\frac{k_1}{m_1} & -\frac{k_2}{m_1} \\
\frac{k_2}{m_2} & -\frac{k_2}{m_2}
\end{bmatrix} \begin{bmatrix}
V_{m1}' \\
V_{m2}'
\end{bmatrix}
\]

we will write this in the form

\[
V_m'' = [A] V_m \quad \text{where } [A] = \text{2x2 matrix}
\]

\[V_m'', V_m \text{ are 2x1}\]
so we have the problem of

\[ V''_m = [A] V_m \]

How can we solve this analytically? Let's "guess" a solution... but keep in mind that \( V_m \) is a vector.

**Guess** \( V_m(t) = \hat{\Lambda} e^{at} \)

\[ V'_m(t) = \hat{\Lambda} \hat{\Lambda} e^{at} \]

\[ V''_m(t) = \hat{\Lambda}^2 \hat{\Lambda} e^{at} \]

If this "guess" solution works, then we need:

\[ V''_m = [A] V_m \]

\[ \hat{\Lambda}^2 \hat{\Lambda} e^{at} = [A] \hat{\Lambda} e^{at} \]

This is the eigenvalue problem.

\[ \{ \begin{align*}
\hat{\Lambda}^2 \hat{\Lambda} & = [A] \hat{\Lambda} \\
\text{or} \\
([A] - \hat{\Lambda}^2 [I]) \hat{\Lambda} & = 0
\end{align*} \}

\[ \rightarrow \text{this form is written in many different ways (i.e. sometimes rather than 'A' people use 'w', 'x', etc...)} \]
Although written in different forms, what it means is the same.

**Given a matrix \( [A] \), can we find scalar quantities \( \lambda_i \) and vectors \( \mathbf{v}_i \) that satisfy this "eigenvalue" problem,

where \( \lambda_i = \text{eigenvalues} \)

\( \mathbf{v}_i = \text{eigen vectors (mode shapes)} \)

So \( ([A] - \lambda^2[I]) \mathbf{v}_i = \mathbf{0} \ldots \) this can only be true if

1) \( \det ([A] - \lambda^2[I]) = 0 \), and then

2) if appropriate \( \mathbf{v}_i \) are selected.

For our problem,

\[
A = \begin{bmatrix}
-\frac{k_1}{m_1} & -\frac{k_2}{m_1} & \frac{k_2}{m_1} \\
\frac{k_2}{m_2} & \frac{k_2}{m_2} & -\frac{k_2}{m_2}
\end{bmatrix}
\]

To make easier, let's assume:

\( k_1 = 3, \ k_2 = 2 \)

\( m_1 = 1, \ m_2 = 1 \)
So \[ A = \begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix} \]

\[ [A] - \lambda^2[I] = \begin{bmatrix} -5 - \lambda^2 & 2 \\ 2 & -2 - \lambda^2 \end{bmatrix} \]

For what value of \( \lambda \) is this determinant equal to zero?

\((-5 - \lambda^2)(-2 - \lambda^2) - 4 = 0\)

\[10 + 5\lambda^2 + 2\lambda^2 + \lambda^4 - 4 = 0\]

\[\lambda^4 + 7\lambda^2 + 6 = 0\]

\[(\lambda^2 + 2)(\lambda^2 + 3) = 0\]

\(\lambda_{2} = \pm \sqrt{3}i, \quad \lambda_{1} = \pm i\)

By convention, we call the smaller value the "first" eigenvalue, etc.

If we use these eigenvalues, what values of \( \gamma \) do we need?
Case 1
\[ \lambda_1 = \pm i, \quad \lambda_2 = -1 \]

\[
(\mathbf{A} - \lambda \mathbf{I}) \mathbf{v} = 0
\]

\[
\begin{bmatrix}
-5 & -1 \\
2 & -2
\end{bmatrix}
\begin{bmatrix}
v_{11} \\
v_{12}
\end{bmatrix}
=
\begin{bmatrix}
0 \\
0
\end{bmatrix}
\]

\[
\begin{bmatrix}
-4 & 2 \\
2 & -1
\end{bmatrix}
\begin{bmatrix}
v_{11} \\
v_{12}
\end{bmatrix}
=
\begin{bmatrix}
0 \\
0
\end{bmatrix}
\]

These are not independent! There are infinite number of combinations of \( v_{11} \) and \( v_{12} \) that will work!

For instance, try \( v_{11} = 1, \ v_{12} = 2 \)
\( v_{12} = 2, \ v_{12} = 4 \)
\( v_{12} = 7, v_{12} = 15.2 \)
\( v_{12} = -2, v_{12} = \sqrt{\frac{17}{3}} \)

They all work! But what is important is the direction, not the magnitude, of the vector.

(Easy way)
To find these ... set \( v_{11} = 1 \) . Then \( v_{12} = 2 \) to solve the matrix equation.

\[
\begin{bmatrix}
v_{11} \\
v_{12}
\end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}
\]

\[ \text{Sometimes this is "normalized" so that magnitude of vector is 1} \]

Case 2
\[ \lambda_2 = \pm \sqrt{6} i, \quad \lambda_2^2 = -6 \]

\((CA) - \lambda^2 [i] \) \(v_2 = 0\)

\[
\begin{bmatrix}
-5 & (-6) \\
2 & -2 & (-6)
\end{bmatrix}
\begin{bmatrix}
v_{21} \\
v_{22}
\end{bmatrix} = \begin{bmatrix} 0 \\
0 \end{bmatrix}
\]

\[
\begin{bmatrix}
1 & 2 \\
2 & 4
\end{bmatrix}
\begin{bmatrix}
v_{11} \\
v_{12}
\end{bmatrix} = \begin{bmatrix} 0 \\
0 \end{bmatrix}
\]

If \( v_{12} = 1 \), then \( v_{22} = -\frac{1}{2} \).

\[ v_2 = \begin{bmatrix} v_{21} \\
v_{22}
\end{bmatrix} = \begin{bmatrix} 0 \\
-\frac{1}{2}
\end{bmatrix} \]

So we have:

Case 1
\[ \lambda_1 = \pm i, \quad v_1 = \begin{bmatrix} 1 \\
0 \end{bmatrix} \]

Case 2
\[ \lambda_2 = \pm \sqrt{6} i, \quad v_2 = \begin{bmatrix} 0 \\
-\frac{1}{2}
\end{bmatrix} \]

These are "eigenvectors" related to the "mode shapes" of the system. The natural frequencies are "normal modes".

Recall that we started by assuming that

\[ v_m(t) = v_i e^{\lambda t} \]

This was our "guess".
So we have
\[ y_m(t) = \left\{ \begin{array}{ll}
1 & \text{if } t < 0 \\
1 & \text{if } t > 0 \\
-1 & \text{if } t = 0
\end{array} \right. \]

Recall from math that we can write these in terms of \( \sin \) and \( \cos \)
\[ y_m(t) = \left\{ \begin{array}{ll}
\frac{1}{2} (a_1 \cos t + b_1 \sin t) + \frac{1}{2} (a_2 \cos \sqrt{6}t + b_2 \sin \sqrt{6}t)
\end{array} \right. \]

where \((a_1, b_1, a_2, b_2)\) are constants depending on the initial conditions. (Why four constants? Because initially 4 state variables... 4 1st order state equations)

MODE 1 \( \left\{ \begin{array}{l}
1 \\
2
\end{array} \right\} f(t) \) • velocities are the same sign • they are moving "in unison" although mass 2 moving twice as fast

MODE 2 \( \left\{ \begin{array}{l}
1 \\
-1/2
\end{array} \right\} f(t) \) • velocities are opposite signs • they are moving in opposite directions, with mass 2 speed 50% that of mass 1.
For an arbitrary set of initial conditions, the system response can be written as a **superposition** of these modes.

For very special cases (very special set of initial conditions), you will only have one mode of behavior.

Let's assume a set of initial conditions and solve for \(a_1, b_1, a_2, b_2\).

\[\text{Ics: } V_{m1}(t=0) = 0 \quad V_{m1}'(t=0) = 3.45\]
\[V_{m2}(t=0) = 0 \quad V_{m2}'(t=0) = -0.775\]

\[V_m(t) = \frac{1}{2} \left( a_1 \cos t + b_1 \sin t \right) + \int_{-\frac{1}{2}}^{\frac{1}{2}} \left( a_2 \cos \omega \sqrt{6} t + b_2 \sin \omega \sqrt{6} t \right) dt\]
\[V_m'(t) = \frac{1}{2} \left( -a_1 \sin t + b_1 \cos t \right) + \int_{-\frac{1}{2}}^{\frac{1}{2}} \sqrt{6} \left( -a_2 \sin \omega \sqrt{6} t + b_2 \cos \omega \sqrt{6} t \right) dt\]

**Note:**
- These are **NOT** accelerations.
- These are **NOT** damping values.
- These are unknown constants.
  (Can also call \(c_1, c_2, c_3, c_4\)
  \(d_1, d_2, d_3, d_4\)
  etc.)
at \( t = 0, \) \( \sin \theta = 0. \) \( \Rightarrow \) plug this in
\( \cos \theta = 1 \) (check if not sure)

\[ \begin{align*}
\vec{v}_m(t=0) \\
\text{ICs:} \\
\{ 0 \} &= \begin{bmatrix} 1 \\ -1/2 \end{bmatrix} a_1 + \begin{bmatrix} 1 \\ 1 \end{bmatrix} a_2 \\
\{ 0 \} &= \begin{bmatrix} 1 & 1 \\ 2 & -1/2 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \\
&\Rightarrow a_1 = 0 \quad a_2 = 0.
\end{align*} \]

\[ \{ 3.45, 0.775 \} = \begin{bmatrix} 1 \\ -1/2 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \]

\[ \begin{align*}
\{ 3.45, 0.775 \} &= \begin{bmatrix} 1 & \sqrt{6} \\ 2 & -\sqrt{6}/2 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \\
&\Rightarrow b_1 = 1 \\
b_2 = 1.
\end{align*} \]

PLUG THESE INTO THE GENERAL SOLUTION

\[ \vec{v}_m(t) = \begin{bmatrix} \vec{v}_m(t) \\ \vec{v}_{m2}(t) \end{bmatrix} = \begin{bmatrix} 1 \\ -1/2 \end{bmatrix} \sin t + \begin{bmatrix} 1 \\ \sqrt{6} \sin \theta \end{bmatrix}. \]