Integrated Cubic Phase Function for Linear FM Signal Analysis

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In this paper, an integrated cubic phase function (ICPF) is introduced for the estimation and detection of linear frequency-modulated (LFM) signals. The ICPF extends the standard cubic phase function (CPF) to handle cases involving low signal-to-noise ratio (SNR) and multi-component LFM signals. The asymptotic mean squared error (MSE) of an ICPF-based estimator as well as the output SNR of an ICPF-based detector are derived in closed form and verified by computer simulation. Comparison with several existing approaches is also included, which shows that the ICPF serves as a good candidate for LFM signal analysis.

I. INTRODUCTION

Frequency-modulated (FM) signals have many applications in radar, sonar, communications, and seismic analysis [1–4]. One important class of such signals are linear FM (LFM) signals frequently encountered in modern radar systems [3, 4]. Due to target motion, radar return signals can be modeled as LFM signals whose parameters, e.g., initial frequencies and chirp-rates, reveal useful information about the target, i.e., its velocity and acceleration.

Detection and parameter estimation of LFM signals have received considerable attention in recent years [5–17]. Early efforts were focused on the analysis of the single-component LFM signal. The maximum likelihood estimator (MLE), which is also called the generalized chirp transform (GCT) in [18], was examined in [5]. Although statistically optimal, the MLE requires a two-dimensional (2-D) joint maximization over the initial frequency and chirp-rate parameters and is computationally demanding. The MLE also requires accurate initial parameter estimates to avoid local maxima, and a numerical search is performed by utilizing a Newton algorithm [5]. Suboptimal techniques are therefore desired for practical implementation. In [6], a phase unwrapping algorithm followed by least-square fitting was proposed, which is suitable for single-component LFM signal estimation at high signal-to-noise ratio (SNR). The discrete polynomial transform (DPT) was employed to reduce the 2-D maximization problem in the MLE to a one-dimensional (1-D) problem [7]. Time-frequency analysis was also studied for LFM signal estimation. For example, the Wigner-Ville distribution (WVD) can be used to track the time-varying frequency of the LFM signal. In general, these techniques can produce good results for single-component LFM signal at moderate to high SNR.

For multi-component LFM signals which arise in many applications, a number of techniques were proposed in [8–13, 16, 17]. The Cramér-Rao bound (CRB) and MLE for multi-component LFM signals were investigated in [10] and [17]. A combined radon-WVD transform (RWT) was proposed to turn the task of tracking straight lines in the time-frequency domain into one of locating the maxima in a 2-D domain [8, 9], which is still computationally complex due to the 2-D optimization. To reduce the 2-D problem to a 1-D problem, the radon-ambiguity transform (RAT) was proposed by exploiting the property that states that auto-terms in the ambiguity function pass through the origin of the ambiguity domain (also see Section III) [13]. It was shown that, compared
with the RWT, the RAT provides comparable performance with reduced computation, especially in cases where the chirp-rate is the only parameter of interest. Nevertheless, the computational complexity remains high since the RAT requires an additional Cartesian-to-polar coordinate transformation and interpolations.

Recently, an instantaneous frequency rate (IFR) estimator using the cubic phase function (CPF) was proposed for FM signal estimation [19]. The CPF-based approach is asymptotically efficient for single-component LFM signals [20]. However, for multi-component LFM signals, the CPF exhibits spurious peaks that cause an identifiability problem. The more LFM components, the higher the probability of spurious peaks [16]. Hence, there is a need to develop robust techniques for the estimation and detection of multi-component LFM signals.

In this paper, an integrated cubic phase function (ICPF) is proposed for LFM signal estimation and detection. The ICPF exploits the property that states that the auto-terms in the CPF are distributed along straight lines parallel to the time axis in the time-“frequency rate” domain, and hence it integrates over these lines to enhance the auto-terms (see Section II for an illustration of the auto-terms and spurious peaks). It is shown that, compared with the CPF, the ICPF can provide considerably lower mean squared error (MSE) at low SNR, lower SNR threshold, and better rejection of spurious peaks or cross-terms for multi-component LFM signals. Additionally, the ICPF is computationally more efficient than the RWT since the former involves a 1-D optimization, as opposed to a 2-D optimization used in the RWT. It is also more efficient than the RAT since the ICPF does not require the computationally expensive Cartesian-to-polar coordinate transformation. The performance of the ICPF is examined in terms of the asymptotic MSE and output SNR.

The paper is organized as follows. The problem formulation is described in Section II. Section III introduces the definition of the ICPF. An ICPF-based parameter estimation for the LFM signal is proposed in Section IV. The asymptotic MSE of the ICPF-based estimates is also included in this section. Section V proposes an ICPF-based detector, and its performance is characterized in terms of the output SNR and the SNR threshold. Numerical examples are provided in Section VI. Finally, conclusions are drawn in Section VII.

II. PROBLEM FORMULATION

Consider noise-contaminated observations of a $K$-component LFM signal:

$$x(n) = \sum_{k=1}^{K} a_k(n) + v(n)$$

$$= \sum_{k=1}^{K} A_k \exp\{j(a_{k,0} + a_{k,1}n + a_{k,2}n^2)\} + v(n)$$

$$n \in \mathbb{Z} \Delta \{n_0, n_0 + 1, \ldots, n_0 + N - 1\}$$ (1)

where $A_k$, $a_{k,0}$, $a_{k,1}$, and $a_{k,2}$ denote the unknown amplitude, phase parameter, frequency parameter, and chirp-rate parameter for the $k$th component, respectively, which are to be estimated, $n_0$ is the initial time index, $N$ is the number of temporal samples, and the noise $v(n)$ is assumed to be a complex white Gaussian noise with zero-mean and variance $\sigma^2$.

Historically, two cases have been considered for the LFM signal energy along straight lines of the CPF. It is noted that the CPF concentrates the probability of spurious peaks [16]. Hence, there is a need to develop robust techniques for the estimation and detection of multi-component LFM signals.

The CPF, which was introduced to extract the IFR [19], is defined as

$$\text{CPF}(n, \Omega) = \sum_{m} x(n + m)x(n - m)e^{-j\Omega m^2}$$

$$m \in \mathbb{L} \Delta \{\lambda : n + \lambda \in \mathbb{Z}, n - \lambda \in \mathbb{Z}\}$$ (3)

where $\Omega$ represents the IFR index for the spectrum of the CPF. It is noted that the CPF concentrates the LFM signal energy along straight lines $\Omega = 2a_{k,2}$ in the $(n - \Omega)$ (time-frequency rate) domain. For comparison, Fig. 1(a)–(c) plots the WVD, ambiguity function (AF), and CPF, respectively, of a 2-component LFM signal with parameters $A_1 = A_2 = 1$, $a_{1,0} = a_{2,0} = 0$, $a_{2,1} = -a_{1,1} = 0.1\pi$, $a_{1,2} = -a_{2,2} = 0.4\pi/N$, and $N = 257$.

The CPF is asymptotically efficient for parameter estimate of a single-component LFM signal estimation [20]. However, an identifiability problem occurs due to cross-terms and spurious peaks when dealing with multi-component LFM signals [16]. For example, consider a 2-component LFM signal

$$x(n) = A_1 \exp\{j(a_{1,0} + a_{1,1}n + a_{1,2}n^2)\}$$

$$+ A_2 \exp\{j(a_{2,0} + a_{2,1}n + a_{2,2}n^2)\}$$ (4)
where the observation noise is ignored for simplicity. The bilinear transform in the CPF results in

\[
x(t + m)x(t - m) = A_1^2 e^{i2\phi_1(n)} e^{i2a_{1,2}m^2} + A_2^2 e^{i2\phi_2(n)} e^{i2a_{2,2}m^2} + A_1A_2 e^{i[\phi_1(n) + \phi_2(n)]} e^{i[(a_{1,2} + a_{2,2})m^2 + \rho(n)m]} + 2A_1A_2 e^{i[\phi_1(n) + \phi_2(n)]} e^{i[(a_{1,2} + a_{2,2})m^2 - \rho(n)m]}
\]

(5)

where

\[
\rho(n) = (a_{1,1} - a_{2,1}) + 2(a_{1,2} - a_{2,2})m.
\]

(6)

The auto-terms in (5) exhibit a quadratic phase in \(m\) with coefficients related to the time index \(n\). By applying the quadratic phase filtering, i.e., \(\sum_m e^{-jm^2}\), the auto-terms are localized along two straight lines independent of the time index, i.e., \(\Omega = 2a_{1,2}\) and \(\Omega = 2a_{2,2}\), while the cross-terms are distributed along trajectories varying with time (see Fig. 1).

However, when \(\rho(n) = 0\) (see (6)), the two hybrid phase terms in \(m\) reduce to quadratic phase terms in \(m\), and (5) reduces to

\[
x(n + m)x(n - m) = A_1^2 e^{i2\phi_1(n)} e^{i2a_{1,2}m^2} + A_2^2 e^{i2\phi_2(n)} e^{i2a_{2,2}m^2} + 2A_1A_2 e^{i[\phi_1(n) + \phi_2(n)]} e^{i[(a_{1,2} + a_{2,2})m^2]}.
\]

(7)
After the quadratic phase filtering, the two cross-terms converge into a single peak at the time index \( n_s \) such that \( \rho(n_s) = 0 \). For the 2-component LFM signal in Fig. 1, \( n_s \approx 32 \). Fig. 1 shows a slice of the CPF at \( n_s = 32 \). It is observed that the highest peak is the spurious peak at \( \Omega = a_{1,2} + a_{2,2} \). The situation becomes worse in dense LFM signal environments. Specifically, for a \( K \)-component LFM signal, there are \((K^2 - K)\) cross-terms which may lead to up to \((K^2 - K)/2\) spurious peaks [16].

III. INTEGRATED CUBIC PHASE FUNCTION

To address the above identifiability problem of the CPF, it is desirable to separate the auto-terms from the cross-terms and spurious peaks. By reviewing (5) and (7), we observe that the auto-terms of the CPF are distributed over straight lines parallel to the time axis, whereas the locations of the cross-terms vary with time, and the spurious peaks occur at discrete locations that are subject to the constraint (6). This property motivates us to integrate along straight lines parallel to the time to enhance the energy of the auto-terms. Once the integral path matches the location of an auto-term, the integral adds up the energy of the auto-terms, thus forming a peak that can be exploited to simplify the detection and estimation of LFM signals.

Specifically, the integral path for the CPF is shown in Fig. 2(c). For comparison, Fig. 2 includes the integral path for the RWT [8] and RAT [13], respectively, where the dashed line shows the integral path that is uniquely determined by the rotation angle \( \theta \), the integral radius \( r \), or both. In particular, the RWT needs to integrate all straight lines in the time-frequency domain by varying the values of both \( \theta \) and \( r \), while the RAT just integrates straight lines passing through the origin of the ambiguity domain (delay-“Doppler frequency” domain, equivalently) by fixing \( r = 0 \) and varying \( \theta \).

In this paper, the ICPF is defined as follows

\[
\text{ICPF}(\Omega) = \sum_n |\text{CPF}(n, \Omega)|^2,
\]

\[
= \sum_n \sum_m \sum_l x(n + m)x(n - m) e^{-j2\pi lm^2 / r^2} e^{-j2\pi \theta l}
\]

where \( m \) and \( l \) are drawn from the set \( L \) defined in (3). By direct substitution of an LFM signal into the above equation, the ICPF exhibits a peak at \( \Omega = 2a_x \). This implies that the detection and parameter estimation of a noisy LFM signal can be performed through a 1-D search of the IFR spectrum. For multi-component LFM signals, the ICPF presents multiple peaks for the auto-terms and suppresses the cross-terms and spurious peaks. The ICPF for the 2-component LFM signal of Fig. 1 is shown in Fig. 1 by a dotted line. It is observed that two distinct peaks corresponding to the auto-terms are shown, and the spurious peak is suppressed.

IV. ICPF-BASED PARAMETER ESTIMATION

As shown in Fig. 1, the locations of the spectrum peaks are proportional to the chirp-rate parameters. Therefore, an ICPF-based estimator for the LFM parameters is introduced in the following.

A. Estimation Algorithm

By searching for peaks in the IFR spectrum, the chirp-rate parameters for the LFM signal can be estimated. For multi-component LFM signals, we can estimate one chirp-rate parameter at a time. Once an estimate of \( a_{k,2} \) is obtained, a dechirping technique is used to convert the observations \( x(n) \) to a sinusoidal signal, and the remaining parameters for the estimated LFM signal are obtained using the following procedure:

1) Dechirp: \( x_d(n) = x(n)e^{-j\hat{\omega}_d n^2} \);
2) Estimate \( a_{k,1} \) by discrete Fourier transform (DFT):
\[
\hat{a}_{k,1} = \arg \max_\omega X_d(\omega)
\]

where \( X_d(\omega) = |\sum_n x_d(n)e^{-j\omega n}|^2 \).
3) Estimate \( a_{k,0} \) and \( A_k \) by least-square: let \( y_d(n) = x_d(n)e^{-j\hat{\omega}_d n^2} \),
\[
\hat{a}_{k,0} = \arg \left\{ \frac{1}{N} \sum_n y_d(n) \right\} = \arg \left\{ \log \frac{1}{N} \sum_n y_d(n) \right\}
\]

where \( \hat{\omega}_d \) and \( a_{k,1} \) are obtained by the first two steps. For 1-component LFM signals, we can estimate the chirp-rate parameter \( a_{k,1} \) by a 1-D search of the IFR spectrum. For multi-component LFM signals, the ICPF presents multiple peaks for the auto-terms and suppresses the cross-terms and spurious peaks.
\[
\hat{A}_k = \frac{1}{N} \sum_n y_d(n) e^{-j\hat{a}_k n} = \exp \left\{ \Re \left( \log \left[ \frac{1}{N} \sum_n y_d(n) \right] \right) \right\}
\]

(11)

where \( \Im \{ \cdot \} \) and \( \Re \{ \cdot \} \) denote the imaginary and real parts of \( \{ \cdot \} \), respectively. The second equality in (10) and (11) shown in [7] will be used in Appendix I for performance analysis.

4) Cancel out the estimated LFM signal using \( x(n) = x(n) - \hat{A}_k e^{j(\hat{a}_k n + \hat{v}_k n + \hat{a}_0 n^2)} \), set \( k = k + 1 \), and repeat steps 1–4 until \( k = K \).

To further improve the estimates, a refining step is helpful to reduce the estimation error caused by interference among different components of the LFM signal. An approach suggested in [11] is adopted here. Specifically, when all parameter estimates are obtained using the above procedure, we reestimate the parameters of each LFM component by canceling out all LFM components other than the one to be estimated and repeating (8) and steps 1–3.

B. Accuracy of the Estimation

In this section, the ICPF-based estimator is examined in terms of its asymptotic bias and MSE. In addition to the chirp-rate parameter estimate, we also study the accuracy of the other parameter estimates, i.e., phase parameter \( \alpha_0 \), frequency parameter \( \alpha_1 \), and amplitude \( A \), which are affected by the \( \alpha_2 \) estimate error when the dechirping technique is used.

An exact analysis of the proposed ICPF-based LFM signal estimator for the multi-component case is difficult due to the interference among different LFM components caused by the nonlinear operation of the ICPF. Note that by estimating one component at a time and interference cancelation (as described in Section III), our estimator effectively converts the problem into a series of single-component LFM signal estimations. In the sequel, we present an analysis for the single-component case, which provides a lower bound on the achievable performance of our estimator, subject to residual interference and imperfect cancelation. A computer simulation shows that the analysis is accurate even for relatively small values of \( N \).

We employ a first-order perturbation analysis similar to the one in [7] for LFM signal estimation. This method is valid for high SNR and for a large number of samples. An SNR threshold effect usually occurs when the high SNR assumption is not met, and the Monte-Carlo simulation can be utilized to verify the theoretical analysis. For the ICPF estimate of the chirp-rate parameter, the first-order perturbation analysis is presented in Appendix IA. The results show that the \( \alpha_2 \) estimate is asymptotically unbiased, i.e., \( E\{\hat{\alpha}_2\} = 0 \), where \( \delta\hat{\alpha}_2 \) denotes the estimation error, and the corresponding asymptotic MSE is

\[
E\{(\delta\hat{\alpha}_2)^2\} = \frac{90}{N^5} \left( \frac{1.008 + 7.433}{\text{SNR}} \right). \tag{12}
\]

Once we have obtained the estimate of the chirp-rate parameter, according to the estimation procedure in Section III, the dechirping technique is applied, and the remaining parameters are estimated using (9), (10), and (11). During this procedure, the error in the \( \alpha_2 \) estimate may propagate to the other estimates, i.e., \( \hat{\alpha}_1, \hat{\alpha}_0 \), and \( \hat{A} \). The error propagation effect is considered here. The derivation of the asymptotic bias and MSE of these estimates is presented in detail in Appendix IB and IC. It is shown that all estimates are asymptotically unbiased. Table I summarizes the asymptotic MSE of these estimates and the corresponding CRBs, which shows that the ICPF-based estimation is asymptotically efficient for the \( \alpha_1 \) and \( A \) estimates, and approximately efficient for the \( \alpha_0 \) and \( \alpha_2 \) estimates at high SNR.

We note that similar observations, i.e., some parameters associated with the LFM signal are asymptotically efficient while the others are not, have been made in other nonlinear LFM signal estimators (see, e.g., [7, 23, 24]). We also note that the MSE of different parameters decreases with \( N \) in different orders. For example, the MSE of \( \hat{\alpha}_2 \) decreases as \( 1/N^5 \) while the MSE of \( \hat{\alpha}_1 \) decreases as \( 1/N^3 \). Only the highest order of \( N \) is counted in each case for the asymptotic analysis.

V. ICPF-BASED DETECTION

LFM signal detection in the presence of noise using the proposed ICPF is considered in this section. Performance analysis of the ICPF-based detector is examined in terms of output SNR as well as the computational complexity.

A. ICPF-Based Detector

Consider the following binary hypothesis testing problem:

\[
H_0: \ x(n) = v(n) \\
H_1: \ x(n) = s(n) + v(n) = A e^{j(\alpha_0 + \alpha_1 n + \alpha_2 n^2)} + v(n) \tag{13}
\]
where the LFM signal under \( H_1 \) has unknown parameters \( A, a_0, a_1, \) and \( a_2, \) and \( v(n) \) is again white Gaussian noise with mean zero and known variance \( \sigma^2. \) A well-known detector for this problem is the generalized likelihood ratio test (GLRT) which substitutes the MLE of the unknown parameters under the alternative hypothesis into the likelihood ratio test [7]

\[
T_{\text{GLR}} = \frac{1}{N} \sum_n x(n)e^{-j\tilde{a}_1, n\omega_0} e^{-j\tilde{a}_2, n\omega_0} H_1 \geq H_0 \gamma_{\text{GLR}} \tag{14}
\]

where \( \tilde{a}_1, \) and \( \tilde{a}_2, \) denote the MLE of the \( a_1 \) and \( a_2, \) respectively, and \( \gamma_{\text{GLR}} \) is the detection threshold which is subject to a specified probability of false alarm. As stated in Section I, the MLE of the \( a_1 \) and \( a_2 \) requires a 2-D grid search and is also subject to a local convergence problem. In [25], it is shown that the GLRT is equivalent to the RWT-based detection, which computes 2-D polar line integrals of the WVD

\[
T_{\text{RWT}} = \max_{\{\rho, \omega\}} \sum_n W(n, \omega_0 + \rho) H_1 \geq H_0 \gamma_{\text{RWT}} \tag{15}
\]

where \( W(n, \omega) \) denotes the WVD of \( x(n), \) and \( \gamma_{\text{RWT}} \) is the RWT-based test threshold.

In practice, it is often the case that the chirp-rate is the only parameter of interest, e.g., detection of a small fast-moving missile launched from a relatively slow moving aircraft [13]. In these cases, the 2-D approach still needs to perform a 2-D search. To simplify the 2-D detection approach, the RAT-based test realizes that the AF of \( x(n) \) is distributed along a line going through the origin of the ambiguity plane and therefore, computes only a 1-D polar line integral

\[
T_{\text{RAT}} = \max_{\{\rho\}} \sum_r |Q(\tau, \rho \tau)|^2 H_1 \geq H_0 \gamma_{\text{RAT}} \tag{16}
\]

where \( Q(n, \omega) \) denotes the ambiguity function of \( x(n) \) and \( \gamma_{\text{RAT}} \) is the RAT-based test threshold. Although the RAT-based test is a 1-D approach, the computation of the RAT still remains high due to the inherent Cartesian-to-polar coordinate transformation and the 2-D interpolation.

In the following, an LFM signal detector, which is computationally more efficient than the above detectors, is introduced by using the proposed ICPF. By recalling that the ICPF concentrates the LFM signal to a peak in the IFR spectrum, we can decide the presence of the LFM signal by searching for peaks in the IFR spectrum exceeding a certain threshold. A ICPF-based detector is thus introduced by simply computing the ICPF and comparing the highest peak with a threshold:

\[
T_{\text{ICPF}} = \max_{\{\Omega\}} \sum_n |\text{ICPF}(n, \Omega)|^2 H_1 \geq H_0 \gamma_{\text{ICPF}} \tag{17}
\]

where \( \gamma_{\text{ICPF}} \) is the corresponding detection threshold. From (17), the ICPF-based test is a 1-D approach as opposed to the 2-D nature of the GLRT/RWT-based test and involves only a 1-D Cartesian line integral, which does not require the Cartesian-to-polar coordinate transformation and the 2-D interpolation of the RAT-based test. As a result, the ICPF-based test appears to be the most computationally efficient approach to detect an LFM signal (see Section VC for more details).

B. Performance Metrics For Detection

In general, the distribution of the test statistic \( T_{\text{ICPF}} \) cannot be obtained in closed form due to the nonlinear operation involved. For practical applications, the histogram of \( T_{\text{ICPF}} \) needs to be estimated from either experimental or simulated data to set the test threshold \( \gamma_{\text{ICPF}}. \) Similarly, the probability of detection cannot be analytically expressed due to the nonlinear transformation. Alternatively, the performance in terms of the probability of detection for a given probability of false alarm is determined by Monte-Carlo simulations, and the results are shown in Section VI.

Another quantity used to characterize the performance of a detector is the output SNR, which is the ratio of the output signal power to the output noise power. In the absence of noise, i.e., \( x(n) = s(n), \) the test statistic of the ICPF-based detector at the maximum point \( \Omega_0 = 2a_2 \) is denoted by \( \text{ICPF}_s(\Omega_0). \) In the presence of noise, \( x(n) = s(n) + v(n), \) the test statistic at \( \Omega_0 \) is a random variable and is denoted as \( \text{ICPF}_s(\Omega_0). \) As a result, the SNR output is defined as [9]

\[
\text{SNR}_{\text{out}} = \frac{|\text{ICPF}_s(\Omega_0)|^2}{\text{var} \{\text{ICPF}_s(\Omega_0)\}} \tag{18}
\]

where \( \{\cdot\} \) denotes the variance of its argument. Here, the input SNR is defined as \( \text{SNR}_{\text{in}} = A^2 / \sigma^2. \) The output SNR for the ICPF-based detector is derived in Appendix II:

\[
\text{SNR}_{\text{out}} = \frac{\text{SNR}_{\text{in}}^3 N^3}{8.1 \text{SNR}_{\text{in}}^2 N^2 + 70.5 N \text{SNR}_{\text{in}} + 192 + 36 \text{SNR}_{\text{in}}^{-1}} \tag{19}
\]

At high input SNR, the above output SNR can be approximated by \( \text{SNR}_{\text{out}} = \text{SNR}_{\text{in}} N / 8.1. \) For comparison, the output SNR of the squared form of the RWT-based and RAT-based detectors were shown in [13, eq. (47)].

C. Computational Complexity

The computational complexity of the RWT-based, RAT-based, and ICPF-based detectors is examined here. Let \( N \) be the number of temporal samples and \( M \) the number of samples in the transformation...
domain where \( M \) is generally chosen larger than \( N \) to help locate the peak [8, 13]. The computational cost of the three detectors is listed in Table II, where RWT\(_1\) denotes the RWT with direct implementation and RWT\(_2\) the one using a dechirping-based implementation [8]. From Table II, the ICPF is seen to be more efficient than the RWT\(_1\) and the RAT since the ICPF avoids the nonlinear Cartesian-to-polar coordinate transformation and the 2-D interpolation which is required in the RWT\(_1\) and RAT. The RWT\(_2\) and ICPF require a similar number of multiplications and additions. However, the ICPF requires only a 1-D maximization compared with the 2-D maximization used by the RWT\(_2\). Note that even if the chirp-rate parameter is the only parameter of interest, the RWT\(_2\) still needs to search a 2-D (chirp-rate and frequency parameters) domain to find the peaks, while the ICPF implements only a 1-D search over the chirp-rate parameter.

D. Threshold Analysis

Based on the output SNR, we can determine the input SNR threshold for the ICPF-based approach. In general, nonlinear estimators often exhibit a threshold effect [26]. That is, at an SNR below a certain threshold, the first-order perturbation analysis, which is based on the assumption of high SNR, is no longer accurate. There are a number of ways to define the SNR threshold (see [7, 23, 24]). From a detection point of view, we define the SNR threshold as the input SNR which results in an output SNR exceeding a preset threshold (e.g., about 13–14 dB for high-resolution radars [1]). With the availability of the output SNR, the SNR threshold of the ICPF-based approach is given by (assuming a preset threshold of 14 dB)

\[
\text{SNR}_{\text{ICPF}}^{\text{ICPF}} = 10 \log_{10} \left( \frac{45}{N} \right). \tag{20}
\]

For a given input SNR, the SNR threshold for the ICPF-based approach decreases as the number of samples increases. The theoretical output SNR in (20) is examined in a number of examples in Section VI.

VI. NUMERICAL EXAMPLES

In this section, numerical examples are provided to illustrate the performance of the proposed methods and verify the theoretical results.

A. Single-Component LFM Signal Estimation

For a single-component LFM signal embedded in white Gaussian noise, the ICPF-based estimates have a lower SNR threshold and smaller MSE than those of the CPF-based estimates. To show this, a single-component LFM signal with parameters \( A_1 = 1, (a_{1,0}, a_{1,1}, a_{1,2}) = (0, 0.2, 0.022\pi/N), n_0 = -(N-1)/2, \) and \( N = 256 \) is considered. Fig. 3 shows the Monte-Carlo simulation results for the \( a_2 \) estimate using the CPF-based estimator, ICPF-based estimator, and the MLE. In simulations, the MLE is implemented in two steps: a coarse 2-D grid search followed by the Newton algorithm [5]. It is seen that the simulated MSE conforms to the theoretical MSE for the ICPF-based estimator. Moreover, the ICPF-based estimate produces lower MSE than the CPF-based estimates at low SNR, especially below -3 dB. The SNR threshold for the ICPF-based estimator is around -8 dB, as predicted by (20), which is 5 dB lower than that of the CPF-based estimator around -3 dB. In addition, it is shown that the ICPF-based estimator has almost identical performance with the MLE above the ICPF-based SNR threshold.

Fig. 4 shows the MSE of the \( a_2 \) estimate with the corresponding CRB [27] as the number of samples \( N \) increases. The other parameters are the same as in the previous example. At SNR = -2 dB, the CPF-based estimator works only when \( N \) is fairly large, while the ICPF-based estimator and the MLE provides lower MSE at smaller \( N \). Moreover, it is seen that the ICPF estimate of \( a_2 \) is nearly efficient for most values of \( N \) considered here, whereas the CPF-based estimator has a noticeable gap to the CRB even after the threshold effect disappears.

With a performance similar to the MLE, the ICPF-based estimator is beneficial from the viewpoint
of complexity. In implementing the MLE, a brute force 2-D grid search is required to locate the convergence region around the optimum. As opposed to the 2-D maximization and search of the MLE, the ICPF-based estimator needs only a 1-D search to estimate the chirp-rate parameter. In addition, as shown in Figs. 3 and 4, once the initial estimate of the MLE is out of the convergence region, the MLE converges to local maxima and achieves worse performance.

B. Multi-Component LFM Signals Estimation

A 2-component LFM signal embedded in complex white Gaussian noise is considered. The parameters of the 2-component LFM signal are $A_1 = A_2 = 1$, $(a_{1,0}, a_{1,1}, a_{1,2}) = (0, 0.2\pi, 0.22\pi/N)$, $(a_{2,0}, a_{2,1}, a_{2,2}) = (0, 0.8\pi, -0.31\pi/N)$, $n_0 = 0$, and $N = 64$ [17]. To reduce estimation bias, as we mentioned in Section III, a refining estimation step is performed by canceling out all other components except the one to be estimated. The MSE of the ICPF-based estimates is seen to match well with the theoretical MSE obtained in Section IV. The MSE is also compared with the CRB corresponding to the case of multi-component LFM signals [10, 17]. The performance of the multi-component MLE was shown in [17].

Fig. 5 shows the MSE for the first component of the LFM signal, using the CPF-based and ICPF-based estimators, respectively. It is shown that the SNR threshold of the ICPF-based estimate is around $-1$ dB, which agrees with the analytical result in (20). At an SNR above the threshold, the MSE for the ICPF-based estimate is close to the theoretical MSE. For the CPF-based estimation, the time index is chosen at $n = 36$, where spurious peaks appear according to $\rho(n) = 0$ in (6). Compared with the ICPF-based estimate, the CPF-based estimate is worse due to interference, including cross-terms and spurious peaks, even at high SNR.

A more challenging case involving a 5-component LFM signal with varying SNRs and close chirp-rates is considered next. Specifically, the $i$th component SNR is defined as $A_i^2/\sigma^2$. The LFM signal parameters are $(A_1, a_{1,0}, a_{1,1}, a_{1,2}) = (1, 0.2\pi, 0.4\pi/N)$, $(A_2, a_{2,0}, a_{2,1}, a_{2,2}) = (0.5, 0.1\pi, 0.1\pi/N)$, $(A_3, a_{3,0}, a_{3,1}, a_{3,2}) = (0.25, 0.4\pi, 0.3\pi/N)$, $(A_4, a_{4,0}, a_{4,1}, a_{4,2}) = (0.125, 0.6\pi, -0.3\pi/N)$, and $(A_5, a_{5,0}, a_{5,1}, a_{5,2}) = (0.0625, 0.3\pi, -0.2\pi/N)$, respectively.

Figs. 6(a) and (b) show the MSE of the chirp-rate estimates for the first and second LFM components, using the CPF-based and the ICPF-based estimators and the MLE, respectively, together with the multi-component cancelation procedure. The SNR shown is the first component’s SNR. For the CPF-based estimation, the time index is chosen at the middle point of observations where no spurious peaks appear. As seen from these figures, the ICPF-based estimator achieves performance similar to the MLE and outperforms the CPF-based estimator. The latter approaches the CRB only at high SNRs.

C. LFM Signal Detection

To compare the performance of the GLRT/RWT-based, RAT-based, and ICPF-based
detectors, an LFM signal is generated using parameters $a_0 = 0.1 \pi$, $a_1 = 0.2 \pi$, $a_2 = 0.1 \pi / N$, and $n_0 = 0$. Due to the nonlinear transformation, analytical expressions of the probability of detection and probability of false alarm cannot be derived in closed form. Here, the simulated probability of detection versus SNR for a given probability of false alarm is shown in Fig. 7, where $N = 64$. It is shown that the ICPF-based detector provides a close detection performance as the GLRT/RWT-based detector, while the RAT-based detector provides worse results which may be attributed to the coordinate transformation and polar line integrals. Specifically, compared with the GLRT/RWT-based detector, the ICPF-based detector shows about 0.5 dB performance loss, but saves in computational complexity from the 2-D search to the 1-D search.

To verify the theoretical output SNR of the ICPF-based detector, an LFM signal with the same parameters as in the above example is simulated, except that $N = 256$. Fig. 8 shows the output SNR obtained from the Monte-Carlo simulation for the three squared-form detectors as well as their theoretical output SNR. It is observed that the simulated output SNR for the ICPF-based detector conforms to the theoretical expression in (19). Moreover, the detection performance for the three detectors are almost the same in terms of the output SNR.

VII. CONCLUSION

The ICPF has been proposed for LFM signal analysis. For either single- or multi-component LFM signals, the ICPF-based approach provides improved estimation accuracy and better capability of rejecting the interference than the CPF-based approach. Performance analysis has been carried out in terms of the asymptotic bias and MSE for the estimation problem and the output SNR and SNR threshold for the detection problem. Comparison with other approaches including the MLE for the estimation and the GLRT for the detection shows that the ICPF provides a reliable and computationally efficient tool for LFM signal detection and estimation.

APPENDIX I. ASYMPTOTIC BIAS AND MSE

A. Chirp-Rate Parameter Estimate

We follow a first-order perturbation analysis as used in [7] for LFM signal estimation. Let $g_0(\Omega)$ be a
noise-free function depending on $\Omega$ and $N$. A random perturbation $\delta g_N(\Omega)$ moves the global maximum $\Omega_0$ of the $g_N(\Omega)$ to the point $\Omega_0 + \delta \Omega$. For the ICPF-based parameter estimator, the random perturbation is due to interference including cross-terms and noise-related terms. To derive the MSE of the ICPF-based estimates, let $g_N(\Omega)$ and $\delta g_N(\Omega)$ be

$$
g_N(\Omega) = \sum_{n} \sum_{m} \sum_{l} s_1 s_2 s_3 s_4 e^{-j\Omega(m^2-l^2)}$$

$$\delta g_N(\Omega) \approx \sum_{n} \sum_{m} \sum_{l} z_{v_1} e^{-j\Omega(m^2-l^2)}$$

where $s_1 = s(n+m)$, $s_2 = s(n-m)$, $s_3 = s(n+l)$, $s_4 = s(n-l)$ for notation simplicity, and $z_{v_1}$ includes the interference with no more than two noise terms due to the high SNR assumption:

$$z_{v_1} \approx s_1 s_2 s_3 s_4 + s_1 s_2 s_3 s_4 + s_1 s_2 s_3 s_4 + s_2 s_3 s_4$$

$$+ \sum_{i=1}^{4} v_i$$

where $\{v_i\}_{i=1}^{4}$ are defined similarly as $\{s_i\}_{i=1}^{4}$ from the noise samples.

By definition of the maximum, we have

$$\frac{\partial[g_N(\Omega) + \delta g_N(\Omega)]}{\partial \Omega} \bigg|_{\Omega_0 + \delta \Omega} = 0.$$ (24)

By using a first-order approximation, the above equation can be approximated as

$$\frac{\partial g_N(\Omega_0)}{\partial \Omega} + \frac{\partial \delta g_N(\Omega_0)}{\partial \Omega} + \frac{\partial^2 g_N(\Omega_0)}{\partial \Omega^2} \delta \Omega \approx 0.$$ (25)

The first term is zero since $\Omega_0$ maximizes $g_N(\Omega)$. Therefore, the estimation error $\delta \Omega$ can be expressed as

$$\delta \Omega = -\frac{\alpha}{\beta}$$ (26)

where $\alpha = \partial g_N(\Omega_0)/\partial \Omega$ and $\beta = \partial^2 g_N(\Omega_0)/\partial \Omega^2$. By using the derivatives of (21) and (22) in the above equation, we have

$$\alpha = \sum_{n} \sum_{m} \sum_{l} (m^2-l^2) z_{v_1} e^{-j\Omega(m^2-l^2)-\pi/2}$$

$$\beta \approx -A^2 \frac{N^7}{630}.$$ (28)

Taking the expectation on (26) yields

$$E\{\delta \Omega\} = -\frac{E\{\alpha\}}{\beta} = 0$$ (29)

due to the fact that

$$E\{\alpha\} \approx -2A^2 \sum_{n} \sum_{m} \sum_{l} (m^2-l^2) [\delta(m-l) + \delta(m+l)]$$

$$= 0$$ (30)

where $\delta(n)$ indicates the Kronecker delta function. Hence, the chirp-rate estimate is asymptotically unbiased as a first-order approximation.

According to (26), the variance of $\delta \Omega$ can be expressed as

$$E\{|\delta \Omega|^2\} = E\{\alpha^2\}/\beta^2.$$ (31)

Based on the high-order moment properties of the Gaussian random variable [28], we have the following types of intermediate results:

$$s_1 s_2 s_3 s_4 e^{-j\Omega(m^2-l^2)-(m^2-l^2)}$$

$$= A^6 \sigma^2 \delta(n-l-n_2 + l_2)$$

$$s_1 s_2 s_3 s_4 e^{-j\Omega(m^2-l^2)-(m^2-l^2)}$$

$$= A^4 \sigma^4 \delta(n-l-n_2 + l_2)$$ (32)

where $s_5 = s(n_2 + m_2)$, $s_6 = s(n_2 - m_2)$, $s_7 = s(n_2 + l_2)$, $s_8 = s(n_2 - l_2)$, and $\{v_i\}_{i=1}^{8}$ are similarly defined.

Based on the above results and (27), $E\{\alpha^2\}$ can be computed as multiple summations of the delta functions in (32), and the results are approximated as

$$E\{\alpha^2\} \approx A^6 \sigma^2 \left( \frac{8N^9}{8744} \right) + A^4 \sigma^4 \left( \frac{4N^8}{1440} + \frac{16N^8}{4033} \right).$$ (33)

Using the above equation in (31), we have

$$E\{|\delta \Omega|^2\} \approx \frac{363}{SNR^3} + \frac{2677}{SNR^2N^6}.$$ (34)

Since $\Omega = 2\alpha_2$, the mean-square value of $\delta \alpha_2$ is

$$E\{|\delta \alpha_2|^2\} \approx \frac{90}{N^3} \left( \frac{1.008}{SNR} + \frac{7.433}{SNR^2N^6} \right).$$ (35)

B. Frequency Parameter Estimate

The dechirped signal can be expressed as

$$x_d(n) = x(n)e^{-j\alpha_2 n^2} = x(n)e^{-j\alpha_2 + j\alpha_2 n^2}$$

$$= [s(n) + v(n)]e^{-j\alpha_2 + j\alpha_2 n^2}$$

$$= [A e^{j\alpha_2 + j\alpha_2 n^2} + v(n)]e^{-j\alpha_2 n^2}.$$ (36)

Since $\alpha_2$ is of order $N^{-1/2}$ (see (35)) and $\alpha_2 n^2$ is of order $N^{-1/2}$ for all $n$, the following approximation holds for large $N$ [7]

$$e^{-j\alpha_2 n^2} \approx 1 - j(\alpha_2 n^2).$$ (37)

By using (37), the dechirped signal can be approximated as

$$x_d(n) = A e^{j\alpha_2 + j\alpha_2 n^2} [v(n)(1 - j(\alpha_2 n^2))e^{-j\alpha_2 n^2} - A e^{j\alpha_2 + j\alpha_2 n^2}].$$ (38)
Once again, we apply a first-order perturbation analysis to the \( a_1 \) estimate, which is the frequency location maximizing the magnitude squared DFT of \( x_d(n) \):

\[
g_N(\omega) = \sum_n A e^{j(m_0+n_1)} e^{-j\omega n} \tag{39}
\]

\[
g_N(\omega) = \sum_n [v(n)(1-j(\delta a_2)n^2)e^{-j(\delta a_2)n^2} - Ae^{j(m_0+n_1)}] e^{-j\omega n}. \tag{40}
\]

The functions \( g_N(\omega) \), \( \delta g_N(\omega) \), and their derivatives at the point of the global maximum \( \omega_0 = a_1 \) are given by

\[
g_N(\omega_0) = Ae^{j(m_0+N)} \tag{41}
\]

\[
\frac{\partial g_N(\omega_0)}{\partial \omega} = -j Ae^{j(m_0)} \sum_n n \approx 0 \tag{42}
\]

\[
\frac{\partial^2 g_N(\omega_0)}{\partial \omega^2} = -A e^{j(m_0)} \sum_n n^2 \approx -A e^{j(m_0)} \frac{N^3}{12} \tag{43}
\]

\[
\delta g_N(\omega_0) = \sum_n [v(n)(1-j(\delta a_2)n^2)e^{-j(\delta a_2)n^2} - j A e^{j(m_0)}(\delta a_2)n^2] \approx \sum_n v(n)(1-j(\delta a_2)n^2)e^{-j(\delta a_2)n^2} - j A e^{j(m_0)}(\delta a_2)n^2 \tag{44}
\]

\[
\frac{\partial \delta g_N(\omega_0)}{\partial \omega} = -j \sum_n [v(n)(1-j(\delta a_2)n^2)e^{-j(\delta a_2)n^2} - j A e^{j(m_0)}(\delta a_2)n^2] \approx -j \sum_n v(n)(1-j(\delta a_2)n^2)e^{-j(\delta a_2)n^2}. \tag{45}
\]

By utilizing the first-order analysis for the complex sinusoid signal in [7] along with the above equations, we obtain

\[
\alpha = 2 \Re \left\{ g_N(\omega_0) \frac{\partial^2 g_N(\omega_0)}{\partial \omega^2} + \frac{\partial g_N(\omega_0)}{\partial \omega} \frac{\partial g_N(\omega_0)}{\partial \omega} \right\} = -A^2 \frac{N^4}{6} \tag{46}
\]

\[
\beta = 2 \Re \left\{ g_N(\omega_0) \frac{\partial g_N(\omega_0)}{\partial \omega} + \frac{\partial g_N(\omega_0)}{\partial \omega} \delta g_N(\omega_0) \right\} \approx 2N \Re \left\{ -j \left( \sum_n ns(n)v^*(n) \right) + \sum_n n^3 s(n)v^*(n)j(\delta a_2) \right\} = 2N(\Im \{\eta\} + \Im \{\gamma\}) \tag{47}
\]

Substituting (26) into the above equations yields the following results:

\[
E\{(\Im \{\eta\})^2\} \approx \frac{A^2 \sigma^2 N^3}{24} \tag{48}
\]

\[
E\{\Im \{\gamma\}\} = 2E\{\eta\gamma\} \approx \sigma^4 \tag{49}
\]

\[
E\{\gamma\gamma\} = 0 \tag{50}
\]

\[
E\{\gamma\gamma^*\} \approx \frac{\sigma^4 N^2}{3}. \tag{51}
\]

Hence,

\[
E\{\beta^2\} = 4N^2[\Im\{\eta\}^2] + E(\Im\{\eta\} \Re\{\gamma\}) \tag{52}
\]

Finally, the asymptotic MSE of the \( a_1 \) estimate is

\[
E\{(\delta a_1)^2\} = \frac{E\{\beta^2\}}{\alpha^2} \approx \frac{6}{N^2 \text{SNR}} \left( 1 + \frac{4}{\text{NSNR}} \right). \tag{53}
\]

C. Phase and Amplitude Parameter Estimates

We now derive the asymptotic MSE of the \( a_0 \) and \( \hat{A} \) estimates using the estimation procedure described in Section III. According to (10) and (11), the dechirping technique is used again. Similar to the approximation used in (37), the dechirped signal can be expressed as

\[
x_{d_2}(n) = A e^{j(m_0)}[1 + A^{-2} s^*(n)v(n)](1 - j(\delta a_1)n - j(\delta a_2)n^2) \approx A e^{j(m_0)}[1 + A^{-2} s^*(n)v(n) - j(\delta a_1)n - j(\delta a_2)n^2]. \tag{54}
\]

Let \( \vartheta = (1/N) \sum_n x_{d_2} \). We have

\[
\log \vartheta = \log \left\{ A e^{j(m_0)} \left[ 1 + \frac{1}{A^2N} \sum_n s^*(n)v(n) - j(\delta a_2)^2 \right] \right\} \tag{55}
\]

Using \( \hat{A} = e^{\Re\{\log \vartheta\}} \) in (11) yields

\[
\log \hat{A} = \log A + \Re \left\{ \frac{1}{A^2N} \sum_n s^*(n)v(n) \right\}. \tag{56}
\]

Since \( \log \hat{A} = \log[1 + A(1 + \delta A/A)] \approx \log A + \delta A/A \) [7], the estimation error of \( A \) can be expressed as

\[
\delta A \approx \frac{1}{NA} \Re \left\{ \sum_n s^*(n)v(n) \right\}. \tag{57}
\]

Then the MSE of the amplitude estimate is

\[
E\{(\delta A)^2\} = \frac{1}{N^2A^2} \frac{A^2 \sigma^2 N}{2} \approx \frac{\sigma^2}{2N}. \tag{58}
\]
Meanwhile, the estimate of $a_0$ can be expressed as
\[
\hat{a}_0 = \Theta \{\log \theta\} = a_0 + \frac{1}{N A^2} \Theta \left\{ \sum_n s'(n) v(n) \right\} - \frac{\delta a_2 N^2}{12}. \tag{59}
\]
Therefore, the MSE of the $a_0$ estimate is
\[
E\{\delta a_0^2\} \approx \frac{1}{2 N \text{SNR}} + \frac{1}{144 N^4} E\{\delta a_2^2\} - 0 \approx \frac{1.13}{\text{NSR}} + \frac{4.65}{N^2 \text{SNR}^2}. \tag{60}
\]

APPENDIX II. OUTPUT SNR ANALYSIS

With signal only, the test statistic in (17) at the maximum point $\Omega_0 = 2a_2$ is
\[
\text{ICPF}_x(\Omega_0) = A^4 \sum_{n=0}^{N-1} \sum_m \sum_l s_1 s_2 s_3 s_4 e^{-j \theta_0 (m^2 - l^2)}
\]
\[
= A^4 \sum_{n=0}^{N-1} \sum_m \sum_l 1
\]
\[
= A^4 N^3 + 2N \approx A^4 N^3 \tag{61}
\]
where we consider the case when the observation time is $n = 0, 1, \ldots, N - 1$ in order to compare it with the output SNR results of the RWT and RAT [9, 13]; $m$ and $l$ are subject to the constraint accordingly, and the last approximation is valid for $N \gg 1$. The output SNR for the case $n = -(N - 1)/2, \ldots, 0, \ldots, (N - 1)/2$ can be similarly obtained. When the signal is corrupted by noise, the expectation of $\text{ICPF}_x(\Omega_0)$ can be expressed by exploiting the moment properties of a complex Gaussian random variable [28]
\[
E\{\text{ICPF}_x(\Omega_0)\}
\]
\[
= \sum_n \sum_m \sum_l (s_1 s_2 s_3 s_4 + s_1 s_2 E\{v_2 v'_2\} + s_1 s_3 E\{v_2 v'_3\})
\]
\[
+ s_1 s_3 E\{v_1 v'_2\} + s_2 s_3 E\{v_1 v'_3\} + E\{v_1 v_2 v'_3 v'_4\}) e^{-j \theta_0 (m^2 - l^2)}. \tag{62}
\]
Using the following results,
\[
s_1 s_2 s_3 s_4 e^{-j \theta_0 (m^2 - l^2)} = A^4
\]
\[
s_1 s_2 E\{v_2 v'_2\} e^{-j \theta_0 (m^2 - l^2)} = A^2 \sigma^2 \delta(m - l)
\]
\[
s_1 s_2 E\{v_2 v'_3\} e^{-j \theta_0 (m^2 - l^2)} = A^2 \sigma^2 \delta(m + l)
\]
\[
s_2 s_3 E\{v_1 v'_2\} e^{-j \theta_0 (m^2 - l^2)} = A^2 \sigma^2 \delta(m + l)
\]
\[
s_2 s_3 E\{v_1 v'_3\} e^{-j \theta_0 (m^2 - l^2)} = A^2 \sigma^2 \delta(m - l)
\]
\[
E\{v_1 v_2 v'_3 v'_4\}) e^{-j \theta_0 (m^2 - l^2)} = \sigma^4 [\delta(m - l) + \delta(m + l)]
\]
we can express (62) as
\[
E\{\text{ICPF}_x(\Omega_0)\}
\]
\[
= \sum_n \sum_m \sum_l \{A^4 + (2A^2 \sigma^2 + \sigma^4)[\delta(m + l) + \delta(m - l)]\}
\]
\[
\approx A^4 N^3 + 2A^2 \sigma^2 N^2 + \sigma^4 N^2. \tag{64}
\]
The second-order moment is
\[
E\{|\text{ICPF}_x(\Omega_0)|^2\}
\]
\[
= \sum_n \sum_m \sum_l \sum_{n_2} \sum_{m_2} \sum_{l_2} \times E\{(s_1 + v_1)(s_2 + l_2)(s_3 + v_3)(s_4 + v_4)(s_5 + (s_6 + v_6))
\]
\[
\times (s_7 + v_7)(s_8 + v_8) e^{j \theta_0 (m_2^2 - l_2^2)} e^{-(m^2 - l^2)}\}. \tag{65}
\]
By using the properties shown in (63), (65) can be simplified as multiple summations of delta functions, and the results are
\[
E\{|\text{ICPF}_x(\Omega_0)|^2\}
\]
\[
= A^8 \frac{N^6}{9} + A^6 \sigma^2 \left( \frac{67}{30} N^5 \right) + A^4 \sigma^4 \left( \frac{2}{3} N^5 + \frac{71}{6} N^4 \right)
\]
\[
+ A^2 \sigma^6 \left( \frac{64}{3} N^3 \right) + \sigma^8 (N^4 + 4N^3). \tag{66}
\]
Combining (62) and (66), the variance can be obtained as
\[
\text{var}\{\text{ICPF}_x(\Omega_0)\} = \frac{9}{10} A^6 \sigma^2 N^5 + \frac{47}{6} A^4 \sigma^4 N^4
\]
\[
+ \frac{64}{3} A^2 \sigma^6 N^3 + 4 \sigma^8 N^3. \tag{67}
\]
Substituting (61) and (67) in (18) yields (19).

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