1. Solve the following initial value problems.

(a) (10 pts)

\[ x \frac{dv}{dx} = \frac{1 + 4v^2}{4v}, \quad v(1) = 0. \]

Solution: The d.e. is separable. We rewrite and integrate.

\[ \int \frac{4v}{1 + 4v^2} dv = \int \frac{1}{x} dx \]

\[ \frac{1}{2} \ln(1 + 4v^2) = \ln x + c \]

From the initial condition, we obtain \( c = 0 \). So the implicit solution can be written in various ways according to taste. Here are a few.

\[ \frac{1}{2} \ln(1 + 4v^2) = \ln x \]

\[ \ln(1 + 4v^2) = 2 \ln x = \ln(x^2) \]

\( (1 + 4v^2) = x^2 \)

(b) (10 pts)

\[ \frac{dy}{dx} + 4xy = 8x, \quad y(0) = 0 \]

Solution: This d.e. is linear. The integrating factor is \( \mu = \exp(\int 4x dx) = \exp(2x^2) \). We multiply by the integrating factor, gather terms and integrate.

\[ e^{2x^2} \left( \frac{dy}{dx} + 4xy \right) = 8xe^{2x^2} \]

\[ \frac{d}{dx} \left( e^{2x^2} y \right) = 8xe^{2x^2} \]

\[ e^{2x^2} y = \int 8xe^{2x^2} dx = 2e^{2x^2} + c \]

Applying the initial condition gives \( 0 = 2 + c \), so \( c = -2 \). The explicit solution is

\[ y = 2 - 2e^{-2x^2} \]
Solution. The equation is not separable. The terms if \( x^2 \) and \( y^3 \) show that it is not linear in either variable. We test for an exact d.e.

\[
M = (2xy - 3x^2) \quad \frac{\partial M}{\partial y} = 2x
\]

\[
N = \left( x^2 - \frac{2}{y^3} \right) \quad \frac{\partial N}{\partial x} = 2x
\]

The cross partial derivatives are equal, so the d.e. is exact. We look for \( F(x,y) \) such that \( F_x = M \) and \( F_y = N \). We start with the first equation.

\[
F_x = M = 2xy - 3x^2
\]

\[
F = \int (2xy - 3x^2) \, dx
\]

\[
= x^2y - x^3 + g(y)
\]

Now, for this \( F \), we match the second condition.

\[
F_y = x^2 + g'(y) = N = \left( x^2 - \frac{2}{y^3} \right)
\]

\[
g'(y) = N = -\frac{2}{y^3}
\]

\[
g(y) = \frac{1}{y^2}
\]

\[
F(x,y) = x^2y - x^3 + \frac{1}{y^2}
\]

Hence the solution to the differential equation is \( F(x,y) = x^2y - x^3 + \frac{1}{y^2} = c \). The initial condition yields \( c = 1 - 1 + 1 = -1 \). Hence, the implicit solution is

\[
x^2y - x^3 + \frac{1}{y^2} = -1.
\]
2. (a) (8 pts) Find a general solution of

\[ \frac{d^2y}{dx^2} + 2 \frac{dy}{dx} + 10y = 0, \quad \text{for } -\infty < x < \infty. \]

Solution: This is a homogeneous linear equation with constant coefficients. The characteristic equation is

\[ r^2 + 2r + 10 = 0 \]

\[ (r + 1)^2 = -9 \]

\[ r + 1 = \pm 3i \]

\[ r = -1 \pm 3i \]

A general solution is

\[ y = e^{-x}(c_1 \cos 3x + c_2 \sin 3x). \]

2(b) (8 pts.) Find a general solution of

\[ t^2 \frac{d^2y}{dt^2} + 3t \frac{dy}{dt} + 5y = 0, \quad \text{for } t > 0. \]

Solution: This homogeneous linear equation is of the Cauchy-Euler (equidimensional) type. The indicial equation is

\[ r(r - 1) + 3r + 5 = 0 \]

\[ r^2 + 2r + 5 = 0 \]

\[ (r + 1)^2 = -4 \]

\[ r + 1 = \pm 2i \]

\[ r = -1 \pm 2i \]

Since \( x^{-1+2i} = x^{-1}x^{2i} = x^{-1}(e^{ln x})^{2i} = x^{-1}(e^{2i\ln x}) \), we may write a general solution as

\[ y = \frac{1}{x}[c_1 \cos(2\ln x) + c_2 \sin(2\ln x)]. \]
2(c) (14 pts.) Use the method of undetermined coefficients to find a general solution of
\[ L[y] = y'' + 5y' + 4y = e^{-t} + 4te^{-2t}. \]

Solution: First, we check the homogeneous equation. The characteristic equation is
\[ p(r) = r^2 + 5r + 4 = (r + 4)(r + 1) = 1. \] Roots are \(-1\) and \(-4\), so \(y_h = c_1 e^{-4t} + c_2 e^{-t}\). Since \(p(-1) = 0\), the solution to \(L[y] = e^{-t}\) is
\[ y_{p_1} = \frac{1}{p'(-1)} te^{-t} = \frac{1}{3} te^{-t}. \]

Now, for \(L[y] = 4te^{-2t}\), we set \(y_{p_2} = (At + B)e^{-2t}\) and substitute into the d.e.
\[ y = (At + B)e^{-2t}, \]
\[ y' = Ae^{-2t} - 2(At + B)e^{-2t} = [-2At + (A - 2B)]e^{-2t}, \]
\[ y'' = -2Ae^{-2t} - 2Ae^{-2t} + 4(At + B)e^{-2t} = [4At + (4B - 4A)]e^{-2t}, \]
\[ L[y] = \{[4At + (4B - 4A)] + 5[-2At + (A - 2B)] + 4(At + B)\}e^{-2t} = \{-2At + (A - 2B)\}e^{-2t} = 4te^{-2t}, \]
\[ -2A = 4 \Rightarrow A = -2, \]
\[ A - 2B = 0 \Rightarrow B = -1, \]
\[ y_{p_2} = (-2t - 1)e^{-2t}. \]

Finally, we combine everything to obtain a general solution.
\[ y = y_h + y_{p_1} + y_{p_2} = c_1 e^{-4t} + c_2 e^{-t} + \frac{1}{3} te^{-t} - (2t + 1)e^{-2t}. \]

3 (a) (15 pts.) Consider the differential equation
\[ L[y] = t^2 \frac{d^2y}{dt^2} + 4t \frac{dy}{dt} + 2y = 4 \ln t, \quad \text{for } t > 1. \]

Solutions to the homogeneous equation are \(y_1(t) = \frac{1}{t}\) and \(y_2(t) = \frac{1}{t^2}\).

(i) Use the Wronskian, \(W[y_1, y_2](t)\), to verify that \(y_1\) and \(y_2\) are linearly independent solutions on the interval \(t > 1\).

Solution:
\[ W[y_1, y_2] = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \frac{1}{t} & \frac{1}{t^2} \\ -\frac{1}{t^2} & -\frac{2}{t^3} \end{vmatrix} \]
\[ = \left(\frac{1}{t}\right) \left(-\frac{2}{t^3}\right) - \left(\frac{1}{t^2}\right) \left(-\frac{1}{t^2}\right) \]
\[ = \frac{-2 + 1}{t^4} = \frac{-1}{t^4}. \]

Since the Wronskian is never zero on the interval \((0, \infty)\), the functions are linearly independent.
(ii) Find a particular solution, $y_p(t)$ satisfying

$$L[y] = t^2 \frac{d^2 y}{dt^2} + 4t \frac{dy}{dt} + 2y = 4 \ln t.$$ 

We use the method of variation of parameters to seek a solution of the form $y = v_1 y_1 + v_2 y_2$ using the given solutions.

$$y_1 v_1' + y_2 v_2' = \frac{1}{t} v_1' + \frac{1}{t^2} v_2' = 0$$

$$y_1' v_1' + y_2' v_2' = \frac{-1}{t^2} v_1' + \frac{-2}{t^3} v_2' = \frac{4 \ln t}{t^2}$$

Multiply the first equation by $t$ and the second equation by $t^2$.

$$v_1' + \frac{1}{t} v_2' = 0$$

$$-v_1' + \frac{-2}{t} v_2' = 4 \ln t$$

Add these and the rest is straightforward.

$$\frac{-1}{t} v_2' = 4 \ln t$$

$$v_2' = -4t \ln t$$

$$v_2 = -4 \left(\frac{1}{2} t^2 \ln t - \frac{1}{4} t^2\right) + c_2$$

$$v_1' = -\frac{1}{t} v_2' = 4 \ln t$$

$$v_1 = 4(t \ln t - t) + c_1$$

$$y_p = 4(t \ln t - t) \frac{1}{t} - 4 \left(\frac{1}{2} t^2 \ln t - \frac{1}{4} t^2\right) \frac{1}{t^2}$$

(iii) Find a general solution to the equation.

Solution:

$$y = [c_1 + 4(t \ln t - t)] \frac{1}{t} + \left[c_2 - 4 \left(\frac{1}{2} t^2 \ln t - \frac{1}{4} t^2\right)\right] \frac{1}{t^2}.$$
3 (b) (15 pts.) Classify each of the following differential equations as linear or nonlinear. If nonlinear, identify all terms that make the equation nonlinear. (In all cases, consider y to be the dependent variable and t the independent variable.) Solution:

<table>
<thead>
<tr>
<th>Equation</th>
<th>Linear/nonlinear</th>
<th>Nonlinear terms</th>
</tr>
</thead>
<tbody>
<tr>
<td>(i) (y'' + 3y' + 4 \sin(y) = 0)</td>
<td>nonlinear</td>
<td>(\sin(y))</td>
</tr>
<tr>
<td>(ii) (y'' + 3y' + 4y = \cos(4t))</td>
<td>linear</td>
<td>-</td>
</tr>
<tr>
<td>(iii) (t^2y'' - ty' + y = \ln(t)/t)</td>
<td>linear</td>
<td>-</td>
</tr>
<tr>
<td>(iv) (e^{-t}dy + (t^2y - \sin(t))dt = 0)</td>
<td>linear</td>
<td>-</td>
</tr>
<tr>
<td>(v) (y'' + 2yy' + 4y = 0)</td>
<td>nonlinear</td>
<td>(yy')</td>
</tr>
<tr>
<td>(vi) (ty' + t^3y^2 = 4t^2)</td>
<td>nonlinear</td>
<td>(y^2)</td>
</tr>
</tbody>
</table>

4. (a) (9 pts.) Find the inverse Laplace transform for

\[ F(s) = \frac{2s + 1}{(s + 2)^3} \]

Solution:

\[
F(s) = \frac{2s + 1}{(s + 2)^3} = \frac{2(s + 2) - 3}{(s + 2)^3} = \frac{2}{(s + 2)^2} + \frac{-3}{(s + 2)^3} 
\]

\[
\mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{\frac{2}{(s + 2)^2}\right\} + \mathcal{L}^{-1}\left\{\frac{-3}{(s + 2)^3}\right\} 
\]

\[= 2te^{2t} - \frac{3}{2}te^{-2t} \]

4 (b) (6pts.) Determine the Laplace transform for \(f(t) = t \sin(3t)\).

Solution:

\[
\mathcal{L}\{\sin(3t)\} = \frac{3}{s^2 + 9} 
\]

\[
\mathcal{L}\{t \sin(3t)\} = (-1) \frac{d}{ds}\left(\frac{3}{s^2 + 9}\right) = (-3) \left(\frac{-1}{(s^2 + 9)^2}\right) 
\]

\[= \frac{6s}{(s^2 + 9)^2} \]
4 (c) (15 pts.) Solve using Laplace Transforms:

\[ L[y] = y'' + 2y' + 5y = te^{-2t}, \quad y(0) = 0, \quad y'(0) = 1. \]

Solution:

\[
\mathcal{L}\{y'' + 2y' + 5y\} = \mathcal{L}\{te^{-2t}\}
\]

\[
s^2Y - sy(0) - y'(0) + 2[sY - y(0)] + 5Y = \frac{1}{(s+2)^2}
\]

\[
(s^2 + 2s + 5)Y - 1 = \frac{1}{(s+2)^2}
\]

\[
Y = \frac{1}{(s+1)^2 + 2^2} + \frac{1}{\left[(s+1)^2 + 2^2\right](s+2)^2}
\]

The first fraction on the right side is simple. We apply the partial fractions technique to the second.

\[
\frac{1}{\left[(s+1)^2 + 2^2\right](s+2)^2} = \frac{A}{(s+2)} + \frac{B}{(s+2)^2} + \frac{C(s+1) + 2D}{(s+1)^2 + 2^2}
\]

Multiplication by the common denominator gives

\[
1 = A(s+2)[(s+1)^2 + 2^2] + B[(s+1)^2 + 2^2] + [C(s+1) + 2D](s+2)^2.
\]

Setting \(s = -2\) gives \(B\).

\[
1 = B[1 + 4] \quad \Rightarrow \quad B = \frac{1}{5}
\]

The coefficient of \(s^3\) is easy to see.

\[
0 = A + C \quad \Rightarrow \quad C = -A
\]

Two more equations can be obtained from \(s = 0\) (the constant term) and \(s = -1\).

\[
1 = A(2)(5) + B(5) + (C + 2D)(4)
\]

\[
1 = A(1)(4) + B(4) + 2D(1)
\]

\(B\) is known and \(C\) can be replaced by \(-A\), so these become

\[
6A + 8D = 0
\]

\[
4A + 2D = \frac{1}{5}
\]

The solution is \(A = \frac{2}{25}\), \(B = \frac{1}{5}\), \(C = -\frac{2}{25}\), \(D = -\frac{3}{50}\).

So

\[
Y = \frac{1}{2} \left(\frac{2}{s + 1)^2 + 2^2}\right) + \frac{2}{25} \left(\frac{1}{s + 2}\right) + \frac{1}{5} \left(\frac{1}{s + 2)^2\right) - \frac{2}{25} \left(\frac{s + 1}{(s + 1)^2 + 2^2}\right) - \frac{3}{50} \left(\frac{2}{s + 1)^2 + 2^2}\right)
\]

\[
y = \frac{1}{2}e^{-t}\sin 2t + \frac{2}{25}e^{-2t} + \frac{1}{5}te^{-2t} - \frac{2}{25}e^{-t}\cos 2t - \frac{3}{50}e^{-t}\sin 2t.
\]
5. (a) (15 pts.) Find the first five non-zero terms of the Fourier cosine series for the function

\[ f(x) = \begin{cases} 0, & 0 < x < \frac{1}{2} \\ x, & \frac{1}{2} < x < 1 \end{cases} \]

Solution:

\[
f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \left( \frac{n\pi}{L} x \right)
\]

We have \( L = 1 \).

\[
a_0 = 2 \int_0^1 f(x) \, dx = 2 \left[ \int_0^{1/2} 0 \, dx + \int_{1/2}^1 x \, dx \right]
\]

\[
= 2 \left[ \left. \frac{1}{2} x^2 \right|_{1/2}^1 \right] = \left( 1 - \frac{1}{4} \right) = \frac{3}{4}
\]

\[
a_n = 2 \int_{1/2}^1 x \cos(n\pi x) \, dx = \frac{2}{(n\pi)^2} \left[ \cos n\pi x + n\pi x \sin n\pi x \right]_{1/2}
\]

\[
= \frac{2}{(n\pi)^2} \left[ \left( \cos n\pi - n\pi \sin n\pi \right) - \left( \cos \frac{n\pi}{2} + \frac{n\pi}{2} \sin \frac{n\pi}{2} \right) \right]
\]

\[
a_1 = \frac{2}{\pi^2} \left[ \cos \pi - \cos \frac{\pi}{2} - \frac{\pi}{2} \sin \frac{\pi}{2} \right] = \frac{2}{\pi^2} \left[ -1 - \frac{\pi}{2} \right]
\]

\[
a_2 = \frac{2}{4\pi^2} \left[ \cos 2\pi - \cos \pi - \frac{2\pi}{2} \sin \pi \right] = \frac{4}{4\pi^2}
\]

\[
a_3 = \frac{2}{9\pi^2} \left[ \cos 3\pi - \cos \frac{3\pi}{2} - \frac{3\pi}{2} \sin \frac{3\pi}{2} \right] = \frac{2}{9\pi^2} \left[ -1 + \frac{3\pi}{2} \right]
\]

\[
a_4 = \frac{2}{16\pi^2} \left[ \cos 4\pi - \cos 2\pi - \frac{4\pi}{2} \sin 2\pi \right] = 0
\]

\[
a_5 = \frac{2}{25\pi^2} \left[ \cos 5\pi - \cos \frac{5\pi}{2} - \frac{5\pi}{2} \sin \frac{5\pi}{2} \right] = \frac{2}{25\pi^2} \left[ -1 - \frac{5\pi}{2} \right]
\]

Finally, we can present the requested terms of the series.

\[
f(x) \sim \frac{3}{8} + \left( \frac{2}{\pi^2} \left[ -1 - \frac{\pi}{2} \right] \right) \cos(\pi x) + \left( \frac{1}{\pi^2} \right) \cos(2\pi x) + \left( \frac{2}{9\pi^2} \left[ -1 + \frac{3\pi}{2} \right] \right) \cos(3\pi x)
\]

\[
+ \left( \frac{2}{25\pi^2} \left[ -1 - \frac{5\pi}{2} \right] \right) \cos(5\pi x) + \cdots
\]

5(b) (10 pts.) To what value does the Fourier series of 5a converge at each of the following points?

(i) \( x = -\frac{3}{4} \quad \frac{3}{4} \) (ii) \( x = 0 \quad 0 \) (iii) \( x = \frac{1}{2} \quad \frac{1}{4} \) (iv) \( x = 1 \quad 1 \) (v) \( x = \frac{3}{2} \quad \frac{1}{4} \)
6 (25 pts.) Consider the following initial-boundary value problem for the heat equation.

\[
\begin{align*}
PDE & \quad u_t = 3u_{xx}, \quad 0 < x < \pi, \quad t > 0 \\
BC1 & \quad u(0,t) = 0, \quad t > 0 \\
BC2 & \quad u_x(\pi,t) = 0, \quad t > 0 \\
IC & \quad u(x,0) = 3\sin \frac{x}{2} - \sin \frac{11x}{2} + 7\sin \frac{19x}{2}, \quad 0 < x < \pi
\end{align*}
\]

6 (a) Let the solution be \( u(x,t) = X(x)T(t) \). Use the method of separation of variables and the boundary conditions to obtain an eigenvalue problem for \( X(x) \) and a differential equation for \( T(t) \).

Separation of Variables:

\[
\begin{align*}
X(x)T'(t) &= 3X''(x)T(t) \\
\frac{X''}{X} &= \frac{T'}{3T} = -\lambda
\end{align*}
\]

Two ordinary differential equations result.

\[
\begin{align*}
X'' + \lambda X &= 0 \\
T' + 3\lambda T &= 0
\end{align*}
\]

The boundary conditions lead to boundary conditions on \( X \).

\[
\begin{align*}
u(0,t) &= X(0)T(t) = 0 \quad \Rightarrow X(0) = 0 \\
u_x(\pi,t) &= X'(\pi)T(t) = 0 \quad \Rightarrow X'(\pi) = 0
\end{align*}
\]

6 (b) Solve the eigenvalue problem.

Solution: The differential equation and boundary conditions for \( X(x) \) in part (a) are the eigenvalue problem. The characteristic equation gives \( r = \pm \sqrt{-\lambda} \). We look at the discriminant being positive, zero or negative.

Case 1. \(-\lambda > 0\) \quad \(-\lambda = \mu^2\).

\[
\begin{align*}
X &= c_1 e^{\mu x} + c_2 e^{-\mu x} \\
X' &= \mu(c_1 e^{\mu x} - c_2 e^{-\mu x}) \\
X(0) &= (c_1 + c_2) = 0 \\
c_1 &= -c_2 \\
X'(\pi) &= \mu c_1 (e^{\pi \mu} + e^{-\pi \mu}) = 0 \\
c_1 &= c_2 = 0
\end{align*}
\]

Case 2 \(-\lambda = 0\)
\[ X = c_1 + c_2 x \]
\[ X' = c_2 \]
\[ X(0) = c_1 = 0 \]
\[ X'(\pi) = c_2 = 0 \]

So \( \lambda = 0 \) is not an eigenvalue.
Case 3 \(-\lambda < 0\) \(-\lambda = -\mu^2\).

\[
X = c_1 \cos \mu x + c_2 \sin \mu x
\]

\[
X' = \mu (-c_1 \sin \mu x + c_2 \cos \mu x)
\]

\[
X(0) = c_1 = 0
\]

\[
X'(\pi) = c_2 \cos (\pi \mu) = 0
\]

So, non-zero solutions require \(\cos (\pi \mu) = 0\). We have

\[
\pi \mu = (2n + 1) \frac{\pi}{2} \quad n = 0, 1, 2, 3, \ldots
\]

\[
\mu_n = \frac{2n + 1}{2} \quad n = 1, 2, 3, \ldots
\]

\[
\lambda_n = \left( \frac{2n + 1}{2} \right)^2 \quad n = 1, 2, 3, \ldots
\]

\[
X_n = c_n \sin \left( \frac{2n + 1}{2} x \right) \quad n = 1, 2, 3, \ldots
\]

6 (c) For each eigenvalue, solve the corresponding differential equation for \(T(t)\). For each eigenvalue give the corresponding solution to the heat equation.

Solution: The d.e. for \(T\).

\[
T' + 3\lambda T = 0
\]

\[
T' + 3 \left( \frac{2n + 1}{2} \right)^2 T = 0
\]

\[
T_n = A_n \exp \left( -3 \left( \frac{2n + 1}{2} \right)^2 t \right)
\]
6 (d) Give the formal series solution to the initial-boundary value problem. Apply the initial condition to determine the coefficients of the series. Give the final solution to the problem.

Solution: We combine the results.

\[ u_n(x, t) = X_n(x)T_n(t) \]

\[ = A_n c_n \exp \left( -3 \left( \frac{2n+1}{2} \right)^2 t \right) \sin \left( \frac{2n+1}{2} x \right) \]

A formal solution is obtained by summing. (The two constants are combined in this step.)

\[ u(x, t) = \sum_{n=0}^{\infty} u_n(x, t) \]

\[ = \sum_{n=0}^{\infty} b_n \exp \left( -3 \left( \frac{2n+1}{2} \right)^2 t \right) \sin \left( \frac{2n+1}{2} x \right) \]

To find the coefficients, we use the initial condition.

\[ u(x, 0) = \sum_{n=0}^{\infty} b_n \sin \left( \frac{2n+1}{2} x \right) \]

\[ = 3 \sin \frac{x}{2} - \sin \frac{11x}{2} + 7 \sin \frac{19x}{2} \]

Matching terms leads to \( b_0 = 3, \ b_5 = -1 \) and \( b_9 = 7 \). All the rest are zero. With this, the solution is

\[ u(x, t) = 3 \exp \left( -3 \left( \frac{1}{1} \right)^2 t \right) \sin \frac{x}{2} - \exp \left( -3 \left( \frac{11}{2} \right)^2 t \right) \sin \frac{11x}{2} + 7 \exp \left( -3 \left( \frac{19}{2} \right)^2 t \right) \sin \frac{19x}{2} \]
7 (a) (15 pts.) Find the power series solution to the initial value problem
\[ y'' + 2xy' - 3y = 0, \quad y(0) = 1, \quad y'(0) = 0. \]

Be sure to give the recurrence relation for the coefficients of the power series. Give the first five nonzero terms of the solution.

Solution:

\[ y = \sum_{n=0}^{\infty} a_n x^n, \quad y' = \sum_{n=1}^{\infty} na_n x^{n-1} \quad y'' = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} \]

We substitute into the differential equation.

\[ \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + 2 \sum_{n=1}^{\infty} na_n x^n - 3 \sum_{n=0}^{\infty} a_n x^n = 0 \]

To combine the series, we adjust the indices. In the first, let \( k = n - 2 \). For the others, replace \( n \) by \( k \).

\[ \sum_{k=0}^{\infty} \{k + 2\}(k + 1)a_{k+2}x^k + 2 \sum_{k=1}^{\infty} ka_kx^k - 3 \sum_{k=0}^{\infty} a_kx^k = 0 \]

The middle series does not have a constant term. We split that out and combine the rest.

\[ (2a_2 - 3a_0) + \sum_{k=1}^{\infty} \{k + 2\}(k + 1)a_{k+2} + (2k - 3)s_k \] \( x^k = 0 \)

From the initial conditions,

\[ a_0 = y(0) - 1. \quad a_1 = y'(0) = 0. \]

Then

\[ 2a_2 - 3a_0 = 0 \Rightarrow a_2 = \frac{3}{2}a_0 = \frac{3}{2} \]

\[ \{k + 2\}(k + 1)a_{k+2} + (2k - 3)s_k = 0, \quad k = 1, 2, 3, \ldots \]

This can be written as the recurrence relation.

\[ a_{k+2} = \frac{3 - 2k}{(k + 2)(k + 1)}a_k, \quad k = 1, 2, 3, \ldots \]

We observe that each coefficient comes from that which is two earlier, so all the odd terms are zero. We need three more non-zero terms.

\[ a_4 = \frac{3 - 4}{4 \cdot 3}a_2 = -\frac{3}{4!} \]

\[ a_6 = -\frac{5}{6 \cdot 5}a_4 = \frac{15}{6!} \]

\[ a_8 = -\frac{9}{8 \cdot 7}a_6 = -\frac{135}{8!} \]

The series solution is

\[ y = 1 + \frac{3}{2}x^2 - \frac{3}{4!}x^4 + \frac{15}{6!}x^6 - \frac{135}{8!}x^8 + \cdots \]
7 (b) (15 pts.) Find the eigenvalues and eigenfunctions for
\[ y'' + \lambda y = 0, \quad 0 < x < 5 \]
\[ y'(0) = 0 \]
\[ y(5) = 0 \]

Be sure to consider the cases \( \lambda < 0, \lambda = 0, \) and \( \lambda > 0. \)

Solution: The characteristic equation is \( r^2 + \lambda = 0. \) Thus \( r = \pm \sqrt{-\lambda}. \) We consider the three cases of the quantity under the radical being positive, zero or negative.

Case 1. \(-\lambda > 0.\) We write \(-\lambda = \mu^2.\) The solution to the d.e is
\[ y = c_1 e^{\mu x} + c_2 e^{-\mu x} \]
\[ y' = \mu(c_1 e^{\mu x} - c_2 e^{-\mu x}) \]

From the boundary conditions,1
\[ y'(0) = \mu(c_1 - c_2) = 0 \]
\[ c_1 = c_2 \]
\[ y(5) = c_1 \left( e^{5\mu} + \frac{1}{e^{5\mu}} \right) = 0 \]
\[ c_1 = c_2 = 0 \]

There is no non-zero solution in this case.

Case 2. \(-\lambda = 0.\) The solution to the d.e is
\[ y = c_1 + c_2 x \]
\[ y' = c_2 \]

From the boundary conditions,
\[ y'(0) = c_2 = 0 \]
\[ y(5) = c_1 = 0 \]

Again, there is no non-zero solution.
Case 3. $-\lambda < 0$ We write $-\lambda - \mu^2$. $r = \pm \sqrt{-\mu^2} = \pm \mu i$. The solution to the d.e. is

$$y = c_1 \cos \mu x + c_2 \sin \mu x$$

$$y' = \mu (-c_1 \sin \mu x + c_2 \cos \mu x)$$

From the boundary conditions,

$$y'(0) = \mu c_2 = 0$$

$$c_2 = 0$$

$$y(5) = c_1 \cos 5 \mu$$

For a non-zero solution, we must have

$$\cos 5 \mu = 0$$

$$5 \mu_n = (2n + 1) \frac{\pi}{2} \quad n = 0, 1, 2, \ldots$$

$$\mu_n = (2n + 1) \frac{\pi}{10} \quad n = 0, 1, 2, \ldots$$

So the eigenvalues ($\lambda_n$) and corresponding eigenfunctions ($y_n$) are

$$\lambda_n = \mu_n^2 = \left(2n + 1\right) \frac{\pi^2}{100} \quad n = 0, 1, 2, \ldots$$

$$y_n = c_n \cos \left(\frac{2n + 1}{10} \pi x\right) \quad n = 0, 1, 2, \ldots$$
Table of Laplace Transforms

<table>
<thead>
<tr>
<th>$f(t)$</th>
<th>$F(s) = \mathcal{L}{f}(s) = \hat{f}(s)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$\frac{1}{s}$</td>
</tr>
<tr>
<td>$t^n$</td>
<td>$\frac{n!}{s^{n+1}}$</td>
</tr>
<tr>
<td>$e^{at}$</td>
<td>$\frac{1}{s-a}$</td>
</tr>
<tr>
<td>$\sin bt$</td>
<td>$\frac{b}{s^2 + b^2}$</td>
</tr>
<tr>
<td>$\cos bt$</td>
<td>$\frac{s}{s^2 + b^2}$</td>
</tr>
<tr>
<td>$e^{at}t^n$</td>
<td>$\frac{n!}{(s-a)^{n+1}}$</td>
</tr>
<tr>
<td>$e^{at}\sin bt$</td>
<td>$\frac{b}{(s-a)^2 + b^2}$</td>
</tr>
<tr>
<td>$e^{at}\cos bt$</td>
<td>$\frac{s-a}{(s-a)^2 + b^2}$</td>
</tr>
<tr>
<td>$e^{at}f(t)$</td>
<td>$\mathcal{L}{f}(s-a)$</td>
</tr>
<tr>
<td>$t^n f(t)$</td>
<td>$(-1)^n \frac{d^n}{ds^n} (\mathcal{L}{f}(s))$</td>
</tr>
</tbody>
</table>

Properties of Laplace Transforms

\[
\begin{align*}
\mathcal{L}\{f + g\} &= \mathcal{L}\{f\} + \mathcal{L}\{g\} \\
\mathcal{L}\{cf\} &= c\mathcal{L}\{f\} \\
\mathcal{L}\{f'\}(s) &= s\mathcal{L}\{f\}(s) - f(0) \\
\mathcal{L}\{f''\}(s) &= s^2\mathcal{L}\{f\}(s) - sf(0) - f'(0)
\end{align*}
\]
# Table of Integrals

<table>
<thead>
<tr>
<th>Integral</th>
<th>Result</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \int \sin^2 x , dx )</td>
<td>( -\frac{1}{2} \cos x \sin x + \frac{1}{2} x + C )</td>
</tr>
<tr>
<td>( \int \cos^2 x , dx )</td>
<td>( \frac{1}{2} \cos x \sin x + \frac{1}{2} x + C )</td>
</tr>
<tr>
<td>( \int x \cos bx , dx )</td>
<td>( \frac{1}{b^2} (\cos bx + bx \sin bx) + C )</td>
</tr>
<tr>
<td>( \int x \sin bx , dx )</td>
<td>( \frac{1}{b^2} (\sin bx - bx \cos bx) + C )</td>
</tr>
<tr>
<td>( \int \sec x , dx )</td>
<td>( \ln</td>
</tr>
<tr>
<td>( \int \ln t , dt )</td>
<td>( t \ln t - t + C )</td>
</tr>
<tr>
<td>( \int t \ln t , dt )</td>
<td>( \frac{1}{2} t^2 \ln t - \frac{1}{4} t^2 + C )</td>
</tr>
<tr>
<td>( \int t^2 \ln t , dt )</td>
<td>( \frac{1}{3} t^3 \ln t - \frac{1}{9} t^3 + C )</td>
</tr>
<tr>
<td>( \int t \ln^2 t , dt )</td>
<td>( \frac{1}{4} t^2 (2 \ln^2 t - 2 \ln t + 1) + C )</td>
</tr>
<tr>
<td>( \int \frac{\ln t}{t} , dt )</td>
<td>( \frac{1}{2} \ln^2 t + C )</td>
</tr>
</tbody>
</table>