1. (a) (8 pts) Solve
\[ \frac{dy}{dx} = \sin x \quad 2e^{-y} \quad y(0) = 0. \]
Solution: It is a separable equation. Separate the variables
\[ 2e^{-y}dy = \sin xdx. \]
Integrate
\[-2e^{-y} = -\cos x + C.\]
Use the initial condition \( y(0) = 0 \)
\[-2 = -1 + C \implies C = -1.\]
Thus, the implicit solution is
\[-2e^{-y} = -\cos x - 1\]
(b) (7 pts) Solve
\[ (2x \cos y + 1)dx + \left(-x^{2} \sin y + 2y\right)dy = 0. \]
Solution: The d.e. is not linear nor separable, check if it is exact. Let
\[ M = 2x \cos y + 1, \quad N = -x^{2} \sin y + 2y. \]
The derivatives
\[ M_{y} = -2x \sin y, N_{x} = -2x \sin y \]
are equal and continuous and so it is exact. Using \( F_{x} = M \)
\[ F = \int (2x \cos y + 1)dx = x^{2} \cos y + x + g(y). \]
Using \( F_{y} = N \)
\[ -x^{2} \sin y + g' = -x^{2} \sin y + 2y \implies g'(y) = 2y. \]
Hence we take \( g(y) \) to be an antiderivative of \(-2y \)
\[ g(y) = y^{2} \implies F(x, y) = x^{2} \cos y + x + y^{2} \]
The implicit solution is
\[ x^{2} \cos y + x + y^{2} = c \]
(c) (10 pts) Find a general solution of
\[ x^{2}y'' + 3xy' + 5y = 0. \]
Solution: This is a Cauchy-Euler (or equi-dimensional) equation. We look for a solution of the form \( y = x^{r} \). Substitution gives
\[ r(r - 1)x^{r} + 3rx^{r} + 5x^{r} = 0 \]
\[ \left(r^{2} + 2r + 5\right)x^{r} = 0 \]
\[ r^{2} + 2r + 1 = -4 \]
\[ (r + 1)^{2} = -4 \]
\[ r = -1 \pm 2i \]
One could also use the quadratic formula to solve the indicial equation. So the solutions come from the real and imaginary parts of one of the complex solutions.

\[ x^r = x^{-1+2i} = x^{-2}e^{2i} \]

\[ = x^{-1}(e^{\ln x})^{2i} = x^{-1}e^{2i\ln x} \]

\[ = x^{-1}[(\cos(2\ln x) + i\sin(2\ln x))] \]

Finally, a general solution of the d.e. is

\[ y = c_1x^{-1}\cos(2\ln x) + c_2\sin(2\ln x). \]

2. (a) (12 pts) Find a general solution of

\[ y'' - 2y' = 4x + 2 - 10\sin x. \]

Solution: First, the auxillary equation \( r^2 - 2r = 0 \) has the roots \( r = 0, 2 \) and the homogeneous equation has a general solution

\[ y_h = C_1 + C_2e^{2x}. \]

- Let us use the method of undetermined coefficients. For \( f_1(x) = 4x + 2 \), we see that \( \alpha = 0 \) is a single root of the auxillary equation, and therefore we take \( y_{p1} = x(Ax + B) \).

Substitute it in the equation with \( f_1 \)

\[ 2A - 2(2Ax + B) = 4x + 2, \]

which leads to

\[ -4A = 4, 2A - 2B = 2 \Rightarrow A = -1, B = -2 \Rightarrow y_{p1} = -x^2 - 2x. \]

- For \( f_2(x) = -10\sin x \) there are two ways to find \( y_{p2} \).

First approach: since \( \beta = i \) is not a root of the auxillary equation we take \( y_{p2} = A\cos x + B\sin x \).

Substitute it into the equation to get

\[ -A\cos x - B\sin x - 2(-A\sin x + B\cos x) = -10\sin x, \]

\[ (-A - 2B)\cos x + (-B + 2A)\sin x = -10\sin x, \]

\[ A = -4, B = 2, \]

\[ y_{p2} = -4\cos x + 2\sin x. \]

Second approach: add the equation with \( f_2(x) \) to its complimentary equation to get

\[ w'' - 2w^{ix}. \]

Since \( \alpha = i \) is not a root of the auxillary equation

\[ w_p = \frac{-10e^{ix}}{i^2 - 2i} = \frac{10}{1 + 2i}e^{ix} \]

\[ = 2(1 - 2i)e^{ix} = 2(1 - 2i)(\cos x + i\sin x). \]

\[ y_{p2} = \text{Im}w_p = -4\cos x + 2\sin x. \]

- Using the superposition principle

\[ y_p = -x^2 - 2x - 4\cos x + 2\sin x. \]

- Hence, a general solution is
\[ y = y_h + y_p \]
\[ C_1 + C_2 e^{2x} = -x^2 - 2x - 4\cos x + 2\sin x. \]
2(b) (13 pts.) Find a general solution of

\[ y'' - 2y' + y = e^x \ln x. \]

Solution: First, the auxiliary equation \( r^2 - 2r + 1 = 0 \) has the double root 0 and the homogeneous equation has a general solution

\[ y_h = C_1 e^x + C_2 xe^x. \]

We cannot use the method of undetermined coefficients to find a particular solution. Let us use the method of variation of parameters. Assume

\[ y_p = v_1(x) e^x + v_2(x) xe^x. \]

Find \( v_1(x) \) and \( v_2(x) \) using the system of equations (we can use direct formulas as well):

\[
\begin{cases}
  v_1' e^x + v_2' xe^x = 0, \\
  v_1' e^x + v_2' (e^x + xe^x) = e^x \ln x.
\end{cases}
\]

Subtract the first equation from the second and divide by \( e^x \) to get \( v_2' = \ln x \) and then from the first equation we obtain \( v_1' = -xv_2' = -x \ln x \). Find antiderivatives

\[
\begin{align*}
  v_1 &= \int -x \ln x dx = -\frac{1}{2} x^2 \ln x + \frac{1}{4} x^2, \\
  v_2 &= \int \ln x dx = x \ln x - x.
\end{align*}
\]

Hence, a particular solution is

\[
y_p = (-\frac{1}{2} x^2 \ln x + \frac{1}{4} x^2)e^x + (x \ln x - x)xe^x
= \frac{1}{2} x^2 \ln x e^x - \frac{3}{4} x^2 e^x.
\]

and a general solution is

\[ y = C_1 e^x + C_2 xe^x + \frac{1}{2} x^2 \ln x e^x - \frac{3}{4} x^2 e^x. \]
3. (a) (10 pts.) Let 

\[ g(t) = \begin{cases} 
    t & \text{for } 0 < t < 1 \\
    e^t & \text{for } 1 < t < \infty
\end{cases} \]

Use the definition of the Laplace transform to find \( \mathcal{L}\{g(t)\} \).

Solution: By definition

\[ \mathcal{L}\{g\} = \int_0^\infty e^{-st} g(t) \, dt = \int_0^1 e^{-st} \, dt + \int_1^\infty e^{-st} \, dt. \]

For the first integral

\[ \int_0^1 e^{-st} \, dt = -\frac{1}{s} e^{-st} \bigg|_0^1 = -\frac{1}{s} e^{-s} - \frac{1}{s^2} e^{-s} + \frac{1}{s^2}. \]

For the second integral

\[ \int_1^\infty e^{-st} \, dt = \lim_{N \to \infty} \int_1^N e^{-st} \, dt = \lim_{N \to \infty} \frac{e^{(1-s)t}}{1-s} \bigg|_1^N \]

\[ = \lim_{N \to \infty} \frac{e^{(1-s)N} - e^{1-s}}{1-s} = \frac{e^{1-s}}{s-1} \text{ for } s > 1. \]

For \( s < 1 \) it diverges, and for \( s = 1 \) it also diverges as \( \int_1^\infty dt \). Hence, for \( s > 1 \)

\[ \mathcal{L}\{g\} = \frac{-(s+1)}{s^2} e^{-s} + \frac{1}{s^2} + \frac{e^{1-s}}{s-1}. \]
(b) (15 pts.) Solve using Laplace Transforms:

\[ y'' - 2y' - 3y = 16e^{-t} \quad y(0) = 1, \quad y'(0) = 3. \]

Solution: Let \( Y = \mathcal{L}\{y\}\). Apply the Laplace transform to the equation

\[ s^2 Y - s - 3 - 2(sY - 1) - 3Y = \frac{16}{s + 1}, \]

\[ (s^2 - 2s - 3)Y = \frac{16}{s + 1} + s + 1, \]

\[ Y = \frac{s^2 + 2s + 17}{(s + 1)^2(s - 3)}. \]

Find the inverse Laplace transform of \( Y \). We have

\[ \frac{s^2 + 2s + 17}{(s + 1)^2(s - 3)} = \frac{A}{s - 3} + \frac{B}{s + 1} + \frac{C}{(s + 1)^2}, \]

\[ s^2 + 2s + 17 = A(s + 1)^2 + B(s - 3)(s + 1) + C(s - 3). \]

Substitute \( s = 3 \) to get

\[ 9 + 6 + 17 = A4^2 \Rightarrow 32 = A16 \Rightarrow A = 2. \]

Substitute \( s = -1 \) to get

\[ 1 - 2 + 17 = -4C \Rightarrow 16 = -4C \Rightarrow C = -4. \]

Compare the coefficients at \( s^2 \)

\[ 1 = A + B \Rightarrow B = -1. \]

Therefore

\[ y = \mathcal{L}^{-1}\{Y\} = \mathcal{L}^{-1}\left\{\frac{2}{s - 3}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s + 1}\right\} - \mathcal{L}^{-1}\left\{\frac{4}{(s + 1)^2}\right\} \]

\[ = 2e^{3t} - e^{-t} - 4te^{-t}. \]
4.) a.) (10 pts.) Use separation of variables, \( u(x, t) = X(x)T(t) \), to find two ordinary differential equations which \( X(x) \) and \( T(t) \) must satisfy to be a solution of

\[
e^{x-t} \frac{\partial^2 u}{\partial x^2} - (x-3)^2 t^5 \frac{\partial u}{\partial t} = 0.
\]

Note: Do not solve these ordinary differential equations.

Solution:

\[
\begin{align*}
\frac{\partial^2 u}{\partial x^2} &= X'' \cdot T \\
\frac{\partial^2 u}{\partial t^2} &= X \cdot T''
\end{align*}
\]

\[
e^{x-t} X'' T - (x-3)^2 t^5 XT'' = 0
\]

The last step is the observation that one side is a function only of \( x \) and the other side is a function only of \( t \) so they must be constant. Any name for the constant may be used. I chose \(-\lambda\) since the next step of often an eigenvalue problem. Taking one at a time produces the two O.D.E.s.

\[
e^x X'' + \lambda (x-3)^2 X = 0
\]

\[
t^5 e^t T'' + \lambda T = 0.
\]

b.) (15 pts.) Find

\[
\mathcal{L}^{-1}\left\{ \frac{2s^3 + 5s^2 + 6s + 7}{(s^2 - 1)(s^2 + 4s + 5)} \right\}.
\]

Solution: After completing the square in the quadratic factor in the denominator, we set up the partial fractions expansion needed.

\[
\begin{align*}
\frac{2s^3 + 5s^2 + 6s + 7}{(s^2 - 1)(s^2 + 4s + 5)} &= \frac{2s^3 + 5s^2 + 6s + 7}{(s^2 - 1)(s^2 + 4s + 4 + 1)} \\
&= \frac{2s^3 + 5s^2 + 6s + 7}{(s+1)(s-1)(s^2 + 4s + 4 + 1)} \\
&= \frac{A}{s+1} + \frac{B}{s-1} + \frac{C(s+2) + D}{(s+2)^2 + 1}
\end{align*}
\]

The numerator of the second fraction could be \( Bs + C \), but that would require some extra algebra to invert the Laplace transform.
We multiply by the common denominator.

\[2s^3 + 5s^2 + 6s + 7 = A(s - 1)\left[(s + 2)^2 + 1\right] + B(s + 1)\left[(s + 2)^2 + 1\right] + C(s + 2) + D(s - 1)(s + 1)\]

Set \(s = -1\).

\[-2 + 5 - 6 + 7 = 4 = A(-2)(2)\]

\[A = -1\]

Set \(s = 1\).

\[2 + 5 + 6 + 7 = 20 = B(2)(10)\]

\[B = 1\]

Set \(s = -2\).

\[-16 + 20 - 12 + 7 = -1 = A(-3) + B(-1) + D(-3)(-1)\]

\[-1 = 3 - 1 + 3D\]

\[D = -1\]

Equate the coefficients of \(s^3\).

\[2 = A + B + C\]

\[C = 2\]

Thus

\[
\frac{2s^3 + 5s^2 + 6s + 7}{(s^2 - 1)(s^2 + 4s + 5)} = \frac{-1}{s + 1} + \frac{1}{s - 1} + \frac{2(s + 2) - 1}{(s + 2)^2 + 1}
\]

\[
\mathcal{L}^{-1}\left\{\frac{2s^3 + 5s^2 + 6s + 7}{(s^2 - 1)(s^2 + 4s + 5)}\right\} = -e^{-t} + e^t + 2e^{-2t}\cos t - e^{-2t}\sin t
\]
5. (a) (15 pts.) Find the first five non-zero terms of the Fourier sine series for the function

\[ f(x) = \begin{cases} 
0 & 0 < x < \pi \\
1 & \pi < x < 2\pi 
\end{cases} \]

Solution:

\[ f(x) = \sum_{k=1}^{\infty} a_k \sin \left( \frac{k\pi x}{L} \right) \]

where

\[ a_k = \frac{2}{L} \int_{0}^{L} f(x) \sin \left( \frac{k\pi x}{L} \right) \, dx, \quad k = 1, 2, 3, \ldots \]

Here \( L = 2\pi \) so

\[ f(x) = \sum_{k=1}^{\infty} a_k \sin \left( \frac{kx}{2} \right) \]

where

\[ a_k = \frac{1}{\pi} \left[ \int_{0}^{\pi} 0 \cdot \sin \left( \frac{kx}{2} \right) \, dx + \int_{\pi}^{2\pi} 1 \cdot \sin \left( \frac{kx}{2} \right) \, dx \right] 
= \frac{1}{\pi} \left( \frac{-2}{k} \right) \left[ \cos \left( \frac{kx}{2} \right) \right]_{\pi}^{2\pi} 
= \frac{-2}{k\pi} \left[ \cos k\pi - \cos \left( \frac{k\pi}{2} \right) \right] \]

Calculating until we have five that are non zero, we obtain

\[ a_1 = \frac{-2}{\pi} \left[ -1 - 0 \right] = \frac{2}{\pi} \]
\[ a_2 = \frac{-2}{2\pi} \left[ 1 + 1 \right] = \frac{-4}{2\pi} \]
\[ a_3 = \frac{-2}{3\pi} \left[ -1 - 0 \right] = \frac{2}{3\pi} \]
\[ a_4 = \frac{-2}{4\pi} \left[ 1 - 1 \right] = 0 \]
\[ a_5 = \frac{-2}{5\pi} \left[ -1 - 0 \right] = \frac{2}{5\pi} \]
\[ a_6 = \frac{-2}{6\pi} \left[ 1 + 1 \right] = \frac{-4}{6\pi} \]

Finally, the Fourier series is

\[ f(x) = \sum_{k=1}^{\infty} a_k \sin \left( \frac{kx}{2} \right) = a_1 \sin \left( \frac{x}{2} \right) + a_2 \sin \left( \frac{2x}{2} \right) + a_3 \sin \left( \frac{3x}{2} \right) + \ldots \]

\[ = \frac{2}{\pi} \sin \left( \frac{x}{2} \right) - \frac{4}{2\pi} \sin \left( \frac{2x}{2} \right) + \frac{2}{3\pi} \sin \left( \frac{3x}{2} \right) + \frac{2}{5\pi} \sin \left( \frac{5x}{2} \right) - \frac{4}{6\pi} \sin \left( \frac{6x}{2} \right) + \ldots \]

5(b) (10 pts.) To what value does the Fourier series of 5a converge at each of the following points?

Solution:

(i) \( x = -\frac{3\pi}{2} \) \quad \( f\left( -\frac{3\pi}{2} \right) = -1 \)

(ii) \( x = 0 \) \quad \( f(0) = 0 \)

(iii) \( x = \pi \) \quad \( f(\pi) = \frac{1}{2} \)
(iv) $x = \frac{3\pi}{2}$ \quad $f\left(\frac{3\pi}{2}\right) = 1$ \quad (v) $x = \frac{5\pi}{2}$ \quad $f\left(\frac{5\pi}{2}\right) = -1$. 
6 (25 pts.) Solve the following initial-boundary value problem.

PDE \[ u_t = 3u_{xx}, \quad 0 < x < 4, \quad t > 0 \]

BCs \[ u_x(0, t) = 0 \quad u_x(4, t) = 0 \]

IC \[ u(x, 0) = \cos\left(\frac{\pi}{2} x\right) - 7 \cos\left(\frac{3\pi}{4} x\right) + 5 \cos\left(\frac{3\pi}{2} x\right) \]

You must derive the solution. Your solution should not have any arbitrary constants in it. Show all steps.

Solution: Separation of Variables:

\[ u(x, t) = X(x)T(t) \]

\[ X(x)T'(t) = 3X''(x)T(t) \]

\[ \frac{X''}{X} = \frac{T'}{3T} = -\lambda \]

Two ordinary differential equations result.

\[ X'' + \lambda X = 0 \]

\[ T' + 3\lambda T = 0 \]

The boundary conditions lead to boundary conditions on \( X \).

\[ u_x(0, t) = X'(0)T(t) = 0 \quad \Rightarrow \quad X'(0) = 0 \]

\[ u_x(4, t) = X'(4)T(t) = 0 \quad \Rightarrow \quad X'(4) = 0 \]

We next solve the resulting eigenvalue problem. The characteristic equation gives \( r = \pm \sqrt{-\lambda} \). We look at the discriminant being positive, zero or negative.

Case 1. \(-\lambda > 0 \quad -\lambda = \mu^2\).

\[ X = c_1 e^\mu + c_2 e^{-\mu x} \]

\[ X'(0) = \mu(c_1 - c_2) = 0 \]

\[ c_1 = c_2 \]

\[ X'(4) = \mu c_1 (e^{4\mu} - e^{-4\mu}) = 0 \]

\[ c_1 = c_2 = 0 \]

Case 2 \(-\lambda = 0\)

\[ X = c_1 + c_2 x \]

\[ X'(0) = c_2 = 0 \]

\[ X'(4) = c_2 = 0 \]

So \( \lambda = 0 \) is an eigenvalue and we will label the corresponding eigenfunction \( X_0 = c_0 \).

Case 3 \(-\lambda < 0 \quad -\lambda = -\mu^2\).
\[ X = c_1 \cos \mu x + c_2 \sin \mu x \]
\[ X' = \mu (-c_1 \sin \mu x + c_2 \cos \mu x) \]
\[ X'(0) = \mu c_2 = 0 \]
\[ c_2 = 0 \]
\[ X'(4) = -c_1 \mu \sin 4\mu = 0 \]

So, non-zero solutions require \( \sin 4\mu = 0 \). We have
\[ 4\mu = n\pi \]
\[ \mu_n = \frac{n\pi}{4} \]
\[ \lambda_n = \left( \frac{n\pi}{4} \right)^2 \]
\[ X_n = c_n \cos \left( \frac{n\pi}{4} x \right) \]

We can combine cases 2 and 3 by adjusting the range of the index.
\[ \mu_n = \frac{n\pi}{4} \]
\[ \lambda_n = \left( \frac{n\pi}{4} \right)^2 \]
\[ X_n = c_n \cos \left( \frac{n\pi}{4} x \right) \]

The d.e. for T.
\[ T' + 3\lambda T = 0 \]
\[ T' + 3 \left( \frac{n\pi}{4} \right)^2 T = 0 \]
\[ T_n = A_n \exp \left( -3 \left( \frac{n\pi}{4} \right)^2 t \right) \]

We combine the results.
\[ u_n(x, t) = X_n(x) T_n(t) \]
\[ = A_n c_n \exp \left( -3 \left( \frac{n\pi}{4} \right)^2 t \right) \cos \left( \frac{n\pi}{4} x \right) \]

A formal solution is obtained by summing. (The two constants are combined in this step.)
\[ u(x, t) = \sum_{n=0}^{\infty} a_n \exp \left( -3 \left( \frac{n\pi}{4} \right)^2 t \right) \cos \left( \frac{n\pi}{4} x \right) \]

To find the coefficients, we use the initial condition.
\[ u(x, 0) = \sum_{n=0}^{\infty} a_n \cos \left( \frac{n\pi}{4} x \right) \]
\[ = \cos \left( \frac{\pi}{2} x \right) - 7 \cos \left( \frac{3\pi}{4} x \right) + 5 \cos \left( \frac{3\pi}{2} x \right) \]

Matching terms leads to \( a_2 = 1 \), \( a_3 = -7 \) and \( a_6 = 5 \). All the rest are zero. With this, the solution is
\[ u(x,t) = \exp\left(-3\left(\frac{\pi}{4}\right)^2 t\right) \cos\left(\frac{2\pi}{4} x\right) - 7 \exp\left(-3\left(\frac{3\pi}{4}\right)^2 t\right) \cos\left(\frac{3\pi}{4} x\right) + 5 \exp\left(-3\left(\frac{6\pi}{4}\right)^2 t\right) \cos\left(\frac{6\pi}{4} x\right) \]
7. (a) (13 pts.) Find a general solution of
\[ y'' + 2y' + y = \frac{e^{-x}}{x^2} \]

Solution: We solve this d.e. by the method of variation of parameters. The characteristic equation is
\[ r^2 + 2r + 1 = (r + 1)^2 = 0. \]

Hence
\[ y_h = c_1 e^{-x} + c_2 xe^{-x}. \]

\[ y_1 = e^{-x}, \quad y_1' = -e^{-x} \]
\[ y_2 = xe^{-x}, \quad y_2' = (1-x)e^{-x} \]

Assuming
\[ y_p = v_1 y_1 + v_2 y_2 \]
we have two equations for \( v_1' \) and \( v_2' \).
\[ e^{-x} v_1' + x e^{-x} v_2' = 0 \] \hspace{1cm} (A)
\[ -e^{-x} v_1' + (1-x)e^{-x} v_2' = \frac{e^{-x}}{x^2} \] \hspace{1cm} (B)

Add these to obtain
\[ e^{-x} v_2' = \frac{e^{-x}}{x^2} \] \hspace{1cm} (C)

\[ v_2' = \frac{1}{x^2} \]
\[ v_2 = -\frac{1}{x} + c_2 \]

Now insert equation (C) into equation (A) to obtain
\[ e^{-x} v_1' + \frac{x e^{-x}}{x^2} = 0 \]
\[ v_1' = -\frac{1}{x} \]
\[ v_1 = -\ln x + c_1 \]

A general solution is
\[ y = (c_1 - \ln x)e^{-x} + \left( c_2 - \frac{1}{x} \right) xe^{-x}. \]
7 (b) (12 pts.) Find the power series solution to
\[ y'' + xy' - 2y = 0 \]
near \( x = 0 \). Be sure to give the recurrence relation for the coefficients of the power series. Indicate the two linearly independent solutions and give the first six nonzero terms of the solution.

Solution:
\[ y = \sum_{n=0}^{\infty} a_n x^n. \]
so
\[ y' = \sum_{n=1}^{\infty} a_n n x^{n-1} \]
and
\[ y'' = \sum_{n=2}^{\infty} a_n (n-1)x^{n-2} \]

The differential equation \( \Rightarrow \)
\[ \sum_{n=2}^{\infty} a_n (n-1)x^{n-2} + \sum_{n=1}^{\infty} a_n nx^{n-1} - \sum_{n=0}^{\infty} 2a_n x^n = 0 \]
In the first series, we set \( k = n - 2 \) (which is the same as \( n = k + 2 \)).
\[ \sum_{k=0}^{\infty} (k+2)(k+1)a_{k+2}x^k + \sum_{n=1}^{\infty} a_n nx^{n-1} - \sum_{n=0}^{\infty} 2a_n x^n = 0 \]
Next, replace \( n \) by \( k \) in the other two series.
\[ \sum_{k=0}^{\infty} (k+2)(k+1)a_{k+2}x^k + \sum_{k=1}^{\infty} a_k k x^k - \sum_{k=0}^{\infty} 2a_k x^k = 0 \]
Observe that the middle series has one less term. We bring out the \( k = 0 \) terms from the first and last series and combine the rest.
\[ (2a_2 - 2a_0) + \sum_{k=1}^{\infty} [(k+2)(k+1)a_{k+2} + (k-2)a_k]x^k = 0 \]
From the first term
\[ a_2 = a_0 \]
From the rest, we obtain the recurrence relation.
\[ (k+2)(k+1)a_{k+2} + (k-2)a_k = 0 \quad k = 1, 2, 3, \ldots \]
\[ a_{k+2} = \frac{2-k}{(k+2)(k+1)} a_k \quad k = 1, 2, 3, \ldots \]
We have three non-zero coefficients \( (a_0, a_1, a_2) \) so we need three more.
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\[
\begin{align*}
  k = 1 & \Rightarrow a_3 = \frac{1}{3 \cdot 2} a_1 \\
  k = 2 & \Rightarrow a_4 = 0 \\
  k = 3 & \Rightarrow a_5 = \frac{-1}{5 \cdot 4} a_3 = \frac{-1}{5!} a_1 \\
  k = 4 & \Rightarrow a_6 = \frac{-2}{6 \cdot 5} a_4 = 0 \\
  k = 5 & \Rightarrow a_7 = \frac{-3}{7 \cdot 6} = \frac{(-1)^2 3 \cdot 1}{7!} a_1 \\
\end{align*}
\]

Now, the solution is

\[
y = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots
\]

\[
= a_0 \left( 1 + x^2 \right) + a_1 \left( x + \frac{1}{3!} x^3 - \frac{1}{5!} x^5 + \frac{3 \cdot 1}{7!} x^7 + \ldots \right)
\]
8 (a) (15 pts.) Find the eigenvalues and eigenfunctions for
\[ y'' + \lambda y = 0 \quad 0 < x < 1 \]
\[ y'(0) = y(1) = 0 \]

Be sure to consider the cases \( \lambda < 0, \lambda = 0, \) and \( \lambda > 0. \)

Solution: The characteristic equation is \( r^2 + \lambda = 0. \) Thus \( r = \pm \sqrt{-\lambda}. \) We consider the three cases of the quantity under the radical being positive, zero or negative.

**Case 1.** \( -\lambda > 0. \) We write \( -\lambda = \mu^2. \) The solution to the d.e is
\[ y = c_1 e^x + c_2 e^{-\mu x} \]
\[ y' = \mu (c_1 e^x - c_2 e^{-\mu x}) \]

From the boundary conditions,
\[ y'(0) = \mu (c_1 - c_2) = 0 \]
\[ c_1 = c_2 \]
\[ y(1) = c_1 (e + \frac{1}{e}) = 0 \]
\[ c_1 = c_2 = 0 \]

There is no non-zero solution in this case.

**Case 2** \( -\lambda = 0 \) The solution to the d.e is
\[ y = c_1 + c_2 x \]
\[ y' = c_2 \]

From the boundary conditions,
\[ y'(0) = c_2 = 0 \]
\[ y(1) = c_1 = 0 \]

Again, there is no non-zero solution.

**Case 3.** \( -\lambda < 0 \) We write \( -\lambda = -\mu^2. \) \( r = \pm \sqrt{-\mu^2} = \pm \mu i. \) The solution to the d.e. is
\[ y = c_1 \cos \mu x + c_2 \sin \mu x \]
\[ y' = \mu (-c_1 \sin \mu x + c_2 \cos \mu x) \]

From the boundary conditions,
\[ y'(0) = \mu c_2 = 0 \]
\[ c_2 = 0 \]
\[ y(1) = c_1 \cos \mu \]

For a non-zero solution, we must have
\[ \cos \mu = 0 \]
\[ \mu_n = (2n + 1) \frac{\pi}{2} \quad n = 0, 1, 2, \ldots \]

So the eigenvalues \( (\lambda_n) \) and corresponding eigenfunctions \( (y_n) \) are
\[ \lambda_n = \mu_n^2 = \left[ (2n + 1) \frac{\pi}{2} \right]^2 \quad n = 0, 1, 2, \ldots \]
\[ y_n = c_n \cos \left( \frac{2n + 1}{2} \pi x \right) \quad n = 0, 1, 2, \ldots \]
8(b) (10 pts.) Solve the initial value problem
\[
\frac{dy}{dx} + \frac{y \tan x}{y^2} \sec x = y(0) = 1
\]
Solution: This is a Bernoulli equation. We write it as
\[
y^2 \frac{dy}{dx} + (\tan x)y^3 = \sec x
\]
Let
\[
v = y^3
\]
\[
\frac{dv}{dx} = 3y^2 \frac{dy}{dx}
\]
The d.e becomes
\[
\frac{1}{3} \frac{dv}{dx} + \tan x v = \sec x
\]
\[
\frac{dv}{dx} + 3 \tan x v = 3 \sec x
\]
This is a linear d.e. The integrating factor is
\[
\mu = e^{ \int \frac{3 \tan x}{dx} } = e^{-3 \ln \cos x} = e^{\ln(\cos x)^{-3}} = (\cos x)^{-3} = \sec^3 x
\]
Multiply by the integrating factor.
\[
\sec^3 x \frac{dv}{dx} + 3 \tan x \sec^3 x v = 3 \sec^4 x
\]
\[
\frac{d}{dx} (v \sec^3 x) = 3 \sec^4 x
\]
\[
(v \sec^3 x) = \tan x + \frac{1}{3} \tan^3 x + C
\]
Multiply by \( \cos^3 x \).
\[
v = y^3 = \sin x \cos^2 x + \frac{1}{3} \sin^3 x + C \cos^3 x
\]
From the initial condition
\[
1 = C
\]
The implicit solution is
\[
y^3 = \sin x \cos^2 x + \frac{1}{3} \sin^3 x + \cos^3 x
\]
Table of Laplace Transforms

<table>
<thead>
<tr>
<th>f(t)</th>
<th>F(s) = \mathcal{L}{f}(s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>\frac{t^{n-1}}{(n-1)!}</td>
<td>\frac{1}{s^n} \quad n \geq 1 \quad s &gt; 0</td>
</tr>
<tr>
<td>e^{at}</td>
<td>\frac{1}{s-a} \quad s &gt; a</td>
</tr>
<tr>
<td>\sin bt</td>
<td>\frac{b}{s^2 + b^2} \quad s &gt; 0</td>
</tr>
<tr>
<td>\cos bt</td>
<td>\frac{s}{s^2 + b^2} \quad s &gt; 0</td>
</tr>
<tr>
<td>e^{at}f(t)</td>
<td>\mathcal{L}{f}(s-a)</td>
</tr>
<tr>
<td>t^n f(t)</td>
<td>(-1)^n \frac{d^n}{ds^n} (\mathcal{L}{f}(s))</td>
</tr>
</tbody>
</table>

Table of Integrals

| \int \sin^2 x dx = - \frac{1}{2} \cos x \sin x + \frac{1}{2} x + C |
| \int \cos^2 x dx = \frac{1}{2} \cos x \sin x + \frac{1}{2} x + C |
| \int x \cos bx dx = \frac{1}{b^2} (\cos bx + bx \sin bx) + C |
| \int x \sin bx dx = \frac{1}{b^2} (\sin bx - bx \cos bx) + C |
| \int \tan u du = - \ln(\cos u) + C |
| \int \tan^2 u du = \tan u - u + C |
| \int \sec u du = \ln(\sec u + \tan u) + C |
| \int \sec^2 u du = \tan u + C |
| \int \sec^3 u du = \frac{1}{2} [\sec u \tan u + \ln(\sec u + \tan u)] + C |
| \int \sec^4 u du = \tan u + \frac{1}{3} \tan^3 u + C |
| \int \ln u du = u \ln u - u + C |
| \int u \ln u du = \frac{1}{2} u^2 \ln u - \frac{1}{4} u^2 + C |
| \int u^2 \ln u du = \frac{1}{3} u^3 \ln u - \frac{1}{9} u^3 + C |
| \int \frac{\ln u}{u} du = \frac{1}{2} \ln^2 u + C |