Ma 221 Homework Solutions Spring 2009  
Due January 15/16, 2009

1.2 p.14 #1, 3, 5, 6, 7, 9, 8

1. (a) Show that \( \Phi(x) = 2x^3 \) is an explicit solution to \( x \frac{dy}{dx} = 3y \) on the interval \((-\infty, \infty)\).

Differentiating \( \Phi(x) \) gives:
\[
\Phi'(x) = 6x^2
\]
Substituting \( \Phi \) and \( \Phi' \) for \( y \) and \( y' \):
\[
x \frac{dy}{dx} = 3y \\
x y' = 3y \\
x(6x^2) = 3(2x^3) \\
6x^2 = 6x^2
\]
This identity is true on \((-\infty, \infty)\) and therefore \( \Phi(x) \) is an explicit solution on \((-\infty, \infty)\).

(b) Show that \( \Phi(x) = e^x - x \) is an explicit solution to \( \frac{dy}{dx} + y^2 = e^{2x} + (1 - 2x)e^x + x^2 - 1 \) on the interval \((-\infty, \infty)\).

Differentiating \( \Phi(x) \) gives:
\[
\frac{d\Phi}{dx} = \frac{d}{dx}(e^x - x) = e^x - 1
\]
Substituting \( \Phi \) and \( \Phi' \) for \( y \) and \( y' \):
\[
\frac{d\Phi}{dx} + \Phi(x)^2 = (e^x - 1) + (e^x - x)^2 = (e^x - 1) + (e^{2x} - 2xe^x + x^2) = e^{2x} + (1 - 2x)e^x + x^2 - 1 \\
e^{2x} + (1 - 2x)e^x + x^2 - 1 = e^{2x} + (1 - 2x)e^x + x^2 - 1
\]
Since both sides of the equation are equal, \( \Phi(x) \) is an explicit solution on \((-\infty, \infty)\).

(c) Show that \( \Phi(x) = x^2 - x^{-1} \) is an explicit solution to \( x^2 \frac{d^2y}{dx^2} = 2y \) on the interval \((-\infty, \infty)\).

Differentiating \( \Phi(x) \) twice gives:
\[
\frac{d\Phi}{dx} = \frac{d}{dx}(x^2 - x^{-1}) = 2x - (-1)x^{-2} = 2x + x^{-2} \\
\frac{d^2\Phi}{dx^2} = \frac{d}{dx}(\frac{d\Phi}{dx}) = \frac{d}{dx}(2x + x^{-2}) = 2 + (-2)x^{-3} = 2(1 - x^{-3})
\]
Therefore
\[
x^2 \frac{d^2\Phi}{dx^2} = x^2 \cdot 2(1 - x^{-3}) = 2(x^2 - x^{-1}) = 2\Phi(x)
\]
Both side of the equation are equal therefore, \( \Phi(x) \) is an explicit solution to the differential equation \( x^2y'' = 2y \) on any interval that does not contain the point \( x = 0 \), in particular, on \((0, \infty)\).
3. Determine whether the function \( y = \sin x + x^2 \) is a solution to the differential equation \( \frac{d^2y}{dx^2} + y = x^2 + 2 \).

Since \( y = \sin x + x^2 \), we have \( y' = \cos x + 2x \) and \( y'' = -\sin x + 2 \). These functions are defined on \((-\infty, \infty)\) Substituting these expressions into the differential equation \( y'' + y = x^2 + 2 \) gives \( y'' + y = -\sin x + 2 + \sin x + x^2 = 2 + x^2 = x^2 + 2 \) for all \( x \) in \((-\infty, \infty)\).

Therefore, \( y = \sin x + x^2 \) is a solution to the differential equation on the interval \((-\infty, \infty)\).

5. Determine whether the function \( y \) is a solution to the given differential equation:

\[
x = \cos 2t
\]

Differentiating \( x(t) = \cos 2t \), we get:

\[
\frac{dx}{dt} = -2 \sin 2t,
\]

\[
\frac{dx}{dt} + tx = -2 \sin 2t + t \cos 2t
\]

which is not equal to \( \sin 2t \). Therefore, \( x(t) \) is not a solution to the given differential equation.

6. Determine whether the function \( y \) is a solution to the given differential equation:

\[
\theta = 2e^{3t} - e^{2t}
\]

Differentiating \( \theta = 2e^{3t} - e^{2t} \), we get

\[
\theta' = 6e^{3t} - 2e^{2t}, \quad \theta'' = 18e^{3t} - 4e^{2t}
\]

\[
\frac{d^2\theta}{dt^2} = -\theta \frac{d\theta}{dt} + 3\theta = -2e^{2t}
\]

Therefore, \( \theta \) is not a solution to the given differential equation.

7. Determine whether the function \( y \) is a solution to the given differential equation:

\[
y = e^{2x} - 3e^{-x}
\]

Differentiating \( y = e^{2x} - 3e^{-x} \), we get

\[
y' = 2e^{2x} + 3e^{-x}, \quad y'' = 4e^{2x} - 3e^{-x}
\]

\[
\frac{d^2y}{dx^2} - \frac{dy}{dx} - 2y = 0.
\]

Therefore, \( y = e^{2x} - 3e^{-x} \) is an explicit solution to the given differential equation.

8. Determine whether the function \( y \) is a solution to the given differential equation:

\[
y = 3 \sin 2x + e^{-x}
\]

Differentiating \( y = 3 \sin 2x + e^{-x} \) twice, we get:

\[
\frac{dy}{dx} = 6 \cos 2x - e^{-x}
\]

\[
\frac{d^2y}{dx^2} = -12 \sin 2x + e^{-x}
\]
\[ y'' + 4y = -12 \sin 2x + e^{-x} + 4(3 \sin 2x + e^{-x}) = -12 \sin 2x + e^{-x} + 12 \sin 2x + 4e^{-x} = 5e^{-x} \]

which is equal to \(5e^{-x}\). Therefore, \(y(t)\) is a solution to the given differential equation.