Ma 221 Homework Solutions Spring 2014
Due January 16, 2014

1.2 p.13-14 #2, 4, 5, 6, 7, 8, 10, 11, 17, 20b, 21b, 22a,b

2. (a) Show that $\Phi(x) = x^2$ is an explicit solution to $x \frac{dy}{dx} = 2y$ on the interval $(-\infty, \infty)$.

Differentiating $\Phi(x)$ gives:

$\Phi'(x) = 2x$

Substituting $\Phi$ and $\Phi'$ for $y$ and $y'$:

$x \frac{dy}{dx} = 2y$

$xy' = 2y$

$x(2x) = 2(x^2)$

$2x^2 = 2x^2$

This identity is true on $(-\infty, \infty)$ and therefore $\Phi(x)$ is an explicit solution on $(-\infty, \infty)$.

(b) Show that $\Phi(x) = e^x - x$ is an explicit solution to $\frac{dy}{dx} + 2y = e^{2x} + (1 - 2x)e^x + x^2 - 1$ on the interval $(-\infty, \infty)$.

Differentiating $\Phi(x)$ gives:

$\frac{d\Phi}{dx} = \frac{d}{dx} (e^x - x) = e^x - 1$

Substituting $\Phi$ and $\Phi'$ for $y$ and $y'$:

$\frac{d\Phi}{dx} + \Phi(x)^2 = (e^x - 1) + (e^x - x)^2 = (e^x - 1) + (e^{2x} - 2xe^x + x^2) = e^{2x} + (1 - 2x)e^x + x^2 - 1$

$e^{2x} + (1 - 2x)e^x + x^2 - 1 = e^{2x} + (1 - 2x)e^x + x^2 - 1$

Since both sides of the equation are equal, $\Phi(x)$ is an explicit solution on $(-\infty, \infty)$.

(c) Show that $\Phi(x) = x^2 - x^{-1}$ is an explicit solution to $x^2 \frac{d^2 y}{dx^2} = 2y$ on the interval $(0, \infty)$.

Differentiating $\Phi(x)$ twice gives:

$\frac{d\Phi}{dx} = \frac{d}{dx} (x^2 - x^{-1}) = 2x - (-1)x^{-2} = 2x + x^{-2}$

$\frac{d^2\Phi}{dx^2} = \frac{d}{dx} (\frac{d\Phi}{dx}) = \frac{d}{dx} (2x + x^{-2}) = 2 + (-2)x^{-3} = 2(1 - x^{-3})$

Therefore

$x^2 \frac{d^2\Phi}{dx^2} = x^2 \cdot 2(1 - x^{-3}) = 2(x^2 - x^{-1}) = 2\Phi(x)$

Both sides of the equation are equal therefore, $\Phi(x)$ is an explicit solution to the differential equation $x^2y'' = 2y$ on any interval that does not contain the point $x = 0$, in particular, on $(0, \infty)$.
4. Determine whether the function \( y = \sin x + x^2 \) is a solution to the differential equation \( \frac{d^2y}{dx^2} + y = x^2 + 2 \).

Since \( y = \sin x + x^2 \), we have \( y' = \cos x + 2x \) and \( y'' = -\sin x + 2 \). These functions are defined on \((-\infty, \infty)\) Substituting these expressions into the differential equation \( y'' + y = x^2 + 2 \) gives \( y'' + y = -\sin x + 2 + \sin x + x^2 = 2 + x^2 = x^2 + 2 \) for all \( x \) in \((-\infty, \infty)\).

Therefore, \( y = \sin x + x^2 \) is a solution to the differential equation on the interval \((-\infty, \infty)\).

5. Determine whether the given function is a solution to the given differential equation:

\[ x = \cos 2t \quad \text{dx} + tx = \sin 2t \]

Differentiating \( x(t) = \cos 2t \), we get:
\[
\frac{dx}{dt} = -2\sin 2t (2) = -2\sin 2t \\
\frac{dx}{dt} + tx = -2\sin 2t + t\cos 2t
\]

which is not equal to \( \sin 2t \). Therefore, \( x(t) \) is not a solution to the given differential equation.

6. Determine whether the given function is a solution to the given differential equation:

\[ \theta = 2e^{3t} - e^{2t} \quad \frac{d^2\theta}{dt^2} - \theta \frac{d\theta}{dt} + 3\theta = -2e^{2t} \]

Differentiating \( \theta = 2e^{3t} - e^{2t} \), we get
\[
\theta' = 6e^{3t} - 2e^{2t} \quad \theta'' = 18e^{3t} - 4e^{2t} \\
\frac{d^2\theta}{dt^2} - \theta \frac{d\theta}{dt} + 3\theta = 18e^{3t} - 4e^{2t} = 18e^{3t} - 4e^{2t} + (6e^{3t} - 2e^{2t}) + 3(2e^{3t} - e^{2t}) = 24e^{3t} - 7e^{2t} + 20e^{2t} - 6\theta e^{3t} \]

\( \neq -2e^{2t} \).

Therefore, \( \theta \) is not a solution to the given differential equation.

7. Determine whether the given function is a solution to the given differential equation:

\[ y = 3\sin 2x + e^{-x} \quad y'' + 4y = 5e^{-x} \]

Differentiating \( y(x) = 3\sin 2x + e^{-x} \) twice, we get:
\[
\frac{dy}{dx} = 6\cos 2x - e^{-x} \quad \frac{d^2y}{dx^2} = -12\sin 2x + e^{-x} \\
\frac{d^2y}{dx^2} + 4y = -12\sin 2x + e^{-x} + 4(3\sin 2x + e^{-x}) = -12\sin 2x + e^{-x} + 12\sin 2x + 4e^{-x} = 5e^{-x}
\]

which is equal to \( 5e^{-x} \). Therefore, \( y(t) \) is a solution to the given differential equation.

8. Determine whether the given function is a solution to the given differential equation:

\[ y = e^{2x} - 3e^{-x} \quad \frac{d^2y}{dy^2} - \frac{dy}{dx} - 2y = 0 \]
Differentiating \( y = e^{2x} - 3e^{-x} \), we get
\[
y' = 2e^{2x} + 3e^{-x}, \quad y'' = 4e^{2x} - 3e^{-x}
\]
\[
\frac{d^2y}{dx^2} - \frac{dy}{dx} - 2y = 4e^{2x} - 3e^{-x} - 2e^{2x} - 3e^{-x} - 2e^{2x} - 3e^{-x} = 0.
\]
Therefore, \( y = e^{2x} - 3e^{-x} \) is an explicit solution to the given differential equation.

In problems 10 and 11, determine whether the given relation is an implicit solution to the given differential equation.

10.) \( x^2 + y^2 = 4, \quad \frac{dy}{dx} = \frac{x}{y} \)

Implicitly differentiate to get:
\[
2x + 2yy' = 0
\]
Solve to get:
\[
y' = -\frac{x}{y}
\]
Therefore, \( x^2 + y^2 = 4 \) is not a solution.

11.) \( e^{xy} + y = x - 1, \quad \frac{dy}{dx} = (e^{-xy} - y)/(e^{-xy} + x) \)

Differentiate implicitly to get:
\[
\frac{d}{dx}(e^{xy} + y) = \frac{d}{dx}(x - 1)
\]
\[
e^{xy} \frac{d}{dx}(xy) + \frac{dy}{dx} = 1
\]
\[
e^{xy}(y + x \frac{dy}{dx}) + \frac{dy}{dx} = 1
\]
\[
\frac{dy}{dx} (e^{xy} + 1) = 1 - e^{xy}y
\]
\[
x e^{xy}y' + ye^{xy} + y' = 1
\]
Now, solve for \( y' \):
\[
y' = (1 - ye^{xy})/(xe^{xy} + 1)
\]
Multiply the expression by \( e^{xy}/e^{xy} \):
\[
y' = (e^{-xy} - y)/(e^{-xy} + x)
\]
Thus, it is a solution.

17.) Show that \( \varphi(x) = Ce^{3x} + 1 \) is a solution to \( \frac{d^3y}{dx^3} + 3 \frac{d^2y}{dx^2} + 2 \frac{dy}{dx} = 0 \) for any choice of the constant \( C \).

Differentiate \( \varphi(x) \) to get:
\[
\varphi'(x) = 3Ce^{3x}
\]
Now substitute \( y \) for \( \varphi \) and \( y' \) for \( \varphi' \) so that:
\[
y' - 3y = 3Ce^{3x} - 3(3e^{3x} + 1) = -3
\]
Simplify to get:
\[
-1 = -1
\]
Which is true for any constant \( C \).

20b) Determine for which values of \( m \) the function \( \varphi(x) = e^{mx} \) is a solution to
\[
\frac{d^3y}{dx^3} + 3 \frac{d^2y}{dx^2} + 2 \frac{dy}{dx} = 0
\]

\[
\varphi(x) = e^{mx}
\]
\[
\varphi'(x) = me^{mx}
\]
\[
\varphi''(x) = m^2e^{mx}
\]
\[ \varphi'''(x) = m^3e^{mx} \]

Now, substitute into the DE to get:

\[ m^3e^{mx} + 3(m^2e^{mx}) + 2(me^{mx}) = 0 \]

\[ m^3 + 3m^2 + 2m = 0 \]

\[ m(m^2 + 3m + 2) = 0 \]

\[ m(m + 2)(m + 1) = 0 \]

\[ m = 0, -1, -2 \]

21b) Determine for which values of \( m \) the function \( \varphi(x) = x^m \) is a solution to

\[ x^2\varphi''' - x\varphi' - 5\varphi = 0 \]

\[ \varphi'(x) = mx^{m-1} \]

\[ \varphi''(x) = m(m - 1)x^{m-2} \]

Substituting into the DE we have

\[ x^2[m(m - 1)x^{m-2}] - x[mx^{m-1}] - 5x^m = 0 \]

or

\[ (m^2 - 2m - 5)x^m = 0 \]

Hence

\[ m = \frac{2 \pm \sqrt{4 + 20}}{2} = 1 \pm \sqrt{6} \]

22.) Verify that the function \( \varphi(x) = c_1e^x + c_2e^{-2x} \) is a solution to the linear equation

\[ \frac{d^2y}{dx^2} + \frac{dy}{dx} - 2y = 0 \]

for any choice of \( c_1 \) and \( c_2 \) so that each of the following initial conditions is satisfied:

(a.) \( y(0) = 2 \), \( y'(0) = 1 \)

(b.) \( y(1) = 1 \), \( y'(1) = 0 \)

\[ \varphi(x) = c_1e^x + c_2e^{-2x} \]

\[ \varphi'(x) = c_1e^x - 2c_2e^{-2x} \]

\[ \varphi''(x) = c_1e^x + 4c_2e^{-2x} \]

Substituting into the DE we get:

\[ c_1e^x + 4c_2e^{-2x} + c_1e^x - 2c_2e^{-2x} - 2(c_1e^x + c_2e^{-2x}) = 0 \]

Which simplifies to

\[ 0 = 0 \]

and is a solution.

(a.) Plugging in the initial conditions we get:

\[ \varphi(0) = c_1 + c_2 = 2 \]

\[ \varphi'(0) = c_1 - 2c_2 = 1 \]

Solving the system, we get \( c_1 = \frac{5}{3} \) and \( c_2 = \frac{1}{3} \).

(b.) Plugging in the initial conditions we get:

\[ \varphi(1) = c_1e + c_2e^{-2} = 1 \]

\[ \varphi'(1) = c_1e - 2c_2e^{-2} = 0 \]
Solving the system, we get $c_1 = \frac{2}{3e}$ and $c_2 = \frac{1}{3e^2}$. 