Ma 221 Homework Solutions
Due date: January 30, 2009

2.6 p.79 #21, 23, 28;
4.2 p.167 #1, 3, 7, 9, 17, 26, 27, 29 (For 27 & 29 you may use the Wronskian.)
(Underlined Problems are not handed in)

2.6 p.79 # 21, 23, 28
For 21, 23 and 28 use the method discussed under "Bernoulli Equations" to solve the problems.

21.)
\[ \frac{du}{dx} + \frac{u}{x} = x^2y^2 \]
This is a Bernoulli Equation with \( n = 2 \)

Let
\[ u = y^{1-n} = y^{1-2} = y^{-1} \]
then \( y = u^{-1} \) \( \Rightarrow \) \( y' = -u^2u' \)

and the last equation becomes
\[ -\frac{1}{u^2} \frac{du}{dx} + \frac{1}{ux} = \frac{x^2}{u^2} \] \( \Rightarrow \) \[ \frac{du}{dx} = \frac{1}{x}u = -x^2 \]

This is a linear equation with \( P(x) = -\frac{1}{x} \)

Find \( \mu(x) = e^{\int Pdx} = e^{\int (-\frac{1}{x})dx} = e^{-\ln|x|} = x^{-1} \)

Multiply through by \( \mu(x) \) to get
\[ \frac{1}{x} \frac{du}{dx} - \frac{1}{x^2}u = -x \] \( \Rightarrow \) \[ \frac{d}{dx} \left( \frac{1}{x}u \right) = -x \]

Integrate to get
\[ \frac{1}{x}u = \frac{x^2}{2} + C_1 \] \( \Rightarrow \) \[ u = -\frac{1}{2}x^3 + C_1x \]

Solve explicitly for \( y \)
\[ y = \frac{1}{\frac{x^3}{2} + C_1x} = \frac{2}{C_1x^3} \]

\( y = 0 \) is also a solution to the original equation. It was lost in the first step when we multiplied by \( u^2 \) (also same as dividing by \( y^2 \)).

23.)
\[ \frac{dy}{dx} = \frac{2y}{x} - x^2y^2 \]

or after dividing by \( y^2 \) and moving the first term on the right to the left we have
\[ y^{-2} \frac{dy}{dx} - \frac{2y^{-1}}{x} = -x^2 \]

Let
\[ v = y^{-1} \]
\[ v' = -y^{-2}y' \]

and the last equation becomes
\[ v' + 2\frac{v}{x} = x^2 \]

This is a first order linear equation in \( v \).
The integrating faction for this equation is
\[ e^{\int \frac{2}{x}dx} = x^2 \]
Multiplying the DE by this gives
\[ x^2 v' + 2xv = \frac{d}{dx}(x^2 v) = x^4 \]
Thus
\[ x^2 v = \frac{1}{5} x^5 + c_1 \]
and
\[ \frac{1}{y} = v = \frac{x^3}{5} + \frac{c_1}{x^2} \]
Therefore
\[ y = \left( \frac{x^3 + 5c_1}{5x^3} \right)^{-1} \]
Letting \( C = 5c_1 \) we have finally
\[ y = \left( \frac{5x^2}{x^5 C} \right) \]
\[ y = 0 \] is also a solution to the original equation. It was lost in the first step when we divided by \( y^2 \).

28.)
\[ \frac{dy}{dx} + y^3x + y = 0 \]
Rewrite the equation as
\[ y' + y = -xy^3 \]
Multiply both sides by \( y^{-3} \) to get
\[ y^{-3}y' + y^{-2} = -x \]
Let
\[ v = y^{-2} \]
\[ v' = -2y^{-3}y' \]
The DE then can be written as
\[ -\frac{y'}{2} + v = -x \]
or
\[ v' - 2v = 2x \]
This is a first order linear equation in \( v \). The integrating factor is
\[ e^{\int -2dx} = e^{-2x} \]
Multiplying the last equation by this leads to
\[ e^{-2x}v' - 2e^{-2x}v = \frac{d}{dx}(e^{-2x}v) = 2xe^{-2x} \]
Integrating gives
\[ e^{-2x}v = \int (2xe^{-2x})dx = -\frac{1}{2}e^{-2x} - xe^{-2x} + c \]
Thus
\[ \frac{1}{y^2} = -\frac{1}{2} - x + ce^{2x} \]
Using SNB to check, we have
\[ y' + y = -xy^3 \], Exact solution is:
\[ \frac{1}{\sqrt{C_1 e^{2x} - x + \frac{1}{2}}} \]
\[ C_1 = -4 \]

4.2 p.167 #1, 3, 7, 9, 17, 26, 27, 29
In problems 1, 3 and 9, find a general solution to the given differential equation.
\[ y'' + 5y' + 6y = 0 \]
The auxiliary equation for this problem is \( r^2 + 5r + 6 = (r + 3)(r + 2) = 0 \), which has roots \( r = -3, r = -2 \).
Thus \( \{e^{-3x}, xe^{-2x}\} \) is a fundamental solution set for this differential equation.
Therefore a general solution is
\[
y(x) = c_1 e^{-3x} + c_2 xe^{-2x}
\]
where \( c_1 \) and \( c_2 \) are arbitrary constants.

3.)
\[ y'' + 8y' + 16y = 0 \]
The auxiliary equation for this problem is \( r^2 + 8r + 16 = (r + 4)(r + 4) = 0 \), which has the repeated root \( r = -4 \).
Thus \( \{e^{-4x}, xe^{-4x}\} \) is a fundamental solution set for this differential equation.
Therefore a general solution is
\[
y(x) = c_1 e^{-4x} + c_2 xe^{-4x}
\]
where \( c_1 \) and \( c_2 \) are arbitrary constants.

7.)
Find a general solution to the given differential equation
\[ 2u'' - 7u' - 4u = 0 \]
The auxiliary equation for this problem is \( 2r^2 - 7r - 4 = 0 \Rightarrow (r + 4)(2r - 1) = 0 \) which has roots \( r = -4, \frac{1}{2} \).
Thus the general solution to the given differential equation is
\[
y(t) = c_1 e^{-4t} + c_2 e^{\frac{1}{2}t}
\]
9.)
Find a general solution to the given differential equation
\[ y'' - y' - 11y = 0 \]
The auxiliary equation for this problem is \( r^2 - r - 11 = 0 \) which has roots \( r = (1 + -\sqrt{1 + 4 * 11})/2 = (1 + -3\sqrt{5})/2 \).
Thus the general solution to the given differential equation is
\[
y(t) = c_1 e^{(1+3\sqrt{5})t/2} + c_2 e^{(1-3\sqrt{5})t/2}
\]
For problem 17, solve the initial value problem.
17.) \[ z'' - 2z' - 2z = 0 \]
\[ z(0) = 0 \]
\[ z'(0) = 3 \]
The auxiliary equation for this problem, \( r^2 - 2r - 2 = 0 \), has roots \( r = 1 + -\sqrt{3} \).
Thus, a general solution is given by \( z(t) = c_1 e^{(1+\sqrt{3})t} + c_2 e^{(1-\sqrt{3})t} \).
Differentiating, we find that \( z'(t) = c_1 (1 + \sqrt{3}) e^{(1+\sqrt{3})t} + c_2 (1 - \sqrt{3}) e^{(1-\sqrt{3})t} \).
Substituting \( z(t) \) and \( z'(t) \) into the initial condition yields the system
\[
z(0) = c_1 + c_2 = 0 \implies c_1 = -c_2, c_2 = -\sqrt{3}/2.
\]
Thus, the solution satisfying the given initial conditions is
26.) Boundary Value Problems
Given that every solution to \(y'' + y = 0\) is of the form \(y(x) = c_1 \cos x + c_2 \sin x\), where \(c_1\) and \(c_2\) are arbitrary constants, show that
(a) there is a unique solution to \(y'' + y = 0\) that satisfies the boundary conditions \(y(0) = 2\) and \(y(\pi/2) = 0\).
(b) there is no solution to \(y'' + y = 0\) that satisfies \(y(0) = 2\) and \(y(\pi) = 0\)
(c) there are infinitely many solutions to \(y'' + y = 0\) that satisfy \(y(0) = 2\) and \(y(\pi) = -2\)

(a) Plugging \(y(0) = 2\) into \(y(x)\)
\[2 = C_1 \cos 0 + C_2 \sin 0\]
Therefore
\[C_1 = 2\]
Plugging \(y(\pi/2) = 0\) into \(y(x)\)
\[0 = 2 \cos \frac{\pi}{2} + C_2 \sin \frac{\pi}{2}\]
Therefore
\[C_2 = 0\]
The solution with these conditions is
\[y(x) = 2 \cos x\]

(b) Plugging \(y(0) = 2\) into \(y(x)\)
\[2 = C_1 \cos 0 + C_2 \sin 0\]
Therefore
\[C_1 = 2\]
Plugging \(y(\pi) = 0\) into \(y(x)\)
\[0 = 2 \cos \pi + C_2 \sin \pi\]
The term with \(C_2\) drops out leaving the equation
\[0 = -2\]
This is not true, and thus there is no solution that satisfies these boundary conditions.

(c) Plugging \(y(0) = 2\) into \(y(x)\)
\[2 = C_1 \cos 0 + C_2 \sin 0\]
Therefore
\[C_1 = 2\]
Plugging \(y(\pi) = -2\) into \(y(x)\)
\[-2 = 2 \cos \pi + C_2 \sin \pi\]
The term with \(C_2\) drops out leaving the equation
\[-2 = -2\]
This is true, and thus there are infinitely many solutions with these boundary conditions.

27.) Determine whether the functions \(y_1(t) = e^{-t} \cos 2t; y_2(t) = e^{-t}\) are linearly independent on the interval \((0, 1)\). (For this problem you may use the Wronskian.)
\[ W[y_1, y_2] = y_1(t)y_2'(t) - y_1'(t)y_2(t) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \]

\[ W[e^{-t}\cos 2t, e^{-t}] = \begin{vmatrix} e^{-t}\cos 2t & e^{-t} \\ -(\cos 2t)e^{-t} - 2(\sin 2t)e^{-t} & -e^{-t} \end{vmatrix} = 2(\sin 2t)e^{2(-t)} \neq 0 \]

Hence these functions are LI.

29.) Determine whether the functions \( y_1(t) = te^{2t}; y_2(t) = e^{2t} \) are linearly independent on the interval \((0, 1)\). (For this problem you may use the Wronskian.)

These two functions are solutions of the DE \( y'' - 4y' + 4y = 0 \) since this equation has the characteristic equation \( r^2 - 4r + 4 = (r - 2)^2 = 0 \) and has the repeated root \( r = 2 \). Thus we may check to Wronskian to see if it is zero or not. Now

\[ W[te^{2t}, e^{2t}] = \begin{vmatrix} te^{2t} & e^{2t} \\ e^{2t} + 2te^{2t} & 2e^{2t} \end{vmatrix} = e^{4t} \neq 0 \]

Hence these functions are LI.