Remark. Since we can use either procedure for finding \( F(x, y) \), it may be worthwhile to consider each of the integrals \( \int M(x, y) \, dx \) and \( \int N(x, y) \, dy \). If one is easier to evaluate than the other, this would be sufficient reason for us to use one method over the other. (The skeptical reader should try solving equation (15) by first integrating \( M(x, y) \).)

**Example 4**
Show that

\[
(16) \quad (x + 3x^3 \sin y) \, dx + (x^4 \cos y) \, dy = 0
\]

is not exact but that multiplying this equation by the factor \( x^{-1} \) yields an exact equation. Use this fact to solve (16).

**SOLUTION**
In equation (16), \( M = x + 3x^3 \sin y \) and \( N = x^4 \cos y \). Because

\[
\frac{\partial M}{\partial y} = 3x^3 \cos y \neq 4x^3 \cos y = \frac{\partial N}{\partial x},
\]

equation (16) is not exact. When we multiply (16) by the factor \( x^{-1} \), we obtain

\[
(17) \quad (1 + 3x^2 \sin y) \, dx + (x^3 \cos y) \, dy = 0.
\]

For this new equation, \( M = 1 + 3x^2 \sin y \) and \( N = x^3 \cos y \). If we test for exactness, we now find that

\[
\frac{\partial M}{\partial y} = 3x^2 \cos y = \frac{\partial N}{\partial x},
\]

and hence (17) is exact. Upon solving (17), we find that the solution is given implicitly by \( x + x^3 \sin y = C \). Since equations (16) and (17) differ only by a factor of \( x \), then any solution to one will be a solution for the other whenever \( x \neq 0 \). Hence the solution to equation (16) is given implicitly by \( x + x^3 \sin y = C \).

In Section 2.5 we discuss methods for finding factors that, like \( x^{-1} \) in Example 4, change inexact equations into exact equations.

**Exercises 2.4**

In Problems 1–8, classify the equation as separable, linear exact, or none of these. Notice that some equations may have more than one classification.

1. \((x^2 y + x^4 \cos x) \, dx - x^3 \, dy = 0\)
2. \((x^3 + 2y) \, dx + x \, dy = 0\)
3. \((xy + x^2) \, dx + (x^2 - y) \, dy = 0\)
4. \(\sqrt{-2y - y^2} \, dx + (3 + 2x - x^2) \, dy = 0\)
5. \(y^2 \, dx + (2xy + \cos y) \, dy = 0\)

6. \(xy \, dx + dy = 0\)
7. \(\theta \, dr + (3r - \theta - 1) \, d\theta = 0\)
8. \([2x + y \cos(xy)] \, dx + [x \cos(xy) - 2y] \, dy = 0\)

In Problems 9–20, determine whether the equation is exact. If it is, then solve it.

9. \((2xy + 3) \, dx + (x^2 - 1) \, dy = 0\)
10. \((2x + y) \, dx + (x - 2y) \, dy = 0\)
11. \((\cos x \cos y + 2x)dx - (\sin x \sin y + 2y)dy = 0\)
12. \((e^x \sin y - 3x^2)dx + (e^x \cos y - y^{-2/3})dy = 0\)
13. \((xy/y)dy + (1 + \ln y)dy = 0\)
14. \(e^y(y - t)dt + (1 + e^y)dy = 0\)
15. \(\cos \theta \, d\theta - (r \sin \theta - e^\theta) \, d\theta = 0\)
16. \((ye^{xy} - 1/y)dx + (xe^{xy} + xy^2)dy = 0\)
17. \((1/y)dx - (3y - x/y^2)dy = 0\)
18. \((2x + y^2 - \cos(x + y))dx + (xy - \cos(x + y) - e^y)dy = 0\)
19. \(\left(\frac{2}{\sqrt{1 - x^2}} + y \cos(xy)\right)dx + \left[x \cos(xy) - y^{-1/3}\right]dy = 0\)

In Problems 21–26, solve the initial value problem.
21. \((1/x + 2y^2)x + (2xy^2 - \cos y)dy = 0\), \(y(1) = \pi\)
22. \((ye^{xy} - 1/y)dx + (xe^{xy} + xy^2)dy = 0\), \(y(1) = 1\)
23. \((e^y + ye^y)dt + (te^y + 2)dy = 0\), \(y(0) = -1\)
24. \((e^x + 1)dt + (e^x - 1)dx = 0\), \(x(1) = 1\)
25. \((y^2 \sin x)dx + (1/x - y/x)dy = 0\), \(y(\pi) = 1\)
26. \((\tan y - 2)dx + (x \sec^2 y + 1/y)dy = 0\), \(y(0) = 1\)

27. For each of the following equations, find the most general function \(M(x, y)\) so that the equation is exact.
   (a) \(M(x, y)dx + (\sec^2 y - x/y)dy = 0\)
   (b) \(M(x, y)dx + (\sin x \cos y - xy - e^{-y})dy = 0\)
28. For each of the following equations, find the most general function \(N(x, y)\) so that the equation is exact.
   (a) \(\left[y \cos(xy) + e^x\right]dx + N(x, y)dy = 0\)
   (b) \(ye^{xy} - 4x^3y + 2)dx + N(x, y)dy = 0\)
29. Consider the equation
   \((y^2 + 2xy)dx - x^2 dy = 0\).
   (a) Show that this equation is not exact.
   (b) Show that multiplying both sides of the equation by \(y^{-2}\) yields a new equation that is exact.
   (c) Use the solution of the resulting exact equation to solve the original equation.
   (d) Were any solutions lost in the process?
30. Consider the equation
   \((5x^2y + 6x^3y^2 + 4xy^3)dx + (2x^3 + 3x^4y + 3xy^2)dy = 0\)
   (a) Show that the equation is not exact.
   (b) Multiply the equation by \(x^ny^m\) and determine values for \(n\) and \(m\) that make the resulting equation exact.
   (c) Use the solution of the resulting exact equation to solve the original equation.
31. Argue that in the proof of Theorem 2 the function \(g\) can be taken as
   \[g(y) = \int_{y_0}^y N(x, t) \, dt - \int_{x_0}^x M(s, t) \, ds\]
   This leads ultimately to the representation
   \[F(x, y) = \int_{y_0}^y N(x, t) \, dt - \int_{x_0}^x M(s, y) \, ds\]
   Evaluate this formula directly with \(x_0 = 0\), \(y_0 = 0\) to rework
   (a) Example 1.
   (b) Example 2.
   (c) Example 3.
32. Orthogonal Trajectories. A geometric problem occurring often in engineering is that of finding a family of curves (orthogonal trajectories) that intersects a given family of curves orthogonally at each point. For example, we may be given the lines of force of an electric field and want to find the equation for the equipotential curves. Consider the fam-