The Directional Derivative and the Gradient

Let \( \Phi(x,y,z) \) be a scalar function with first partial derivatives \( \Phi_x, \Phi_y, \) and \( \Phi_z \) in some region of \( x,y,z \)-space. Let \( \vec{r} = xi + yj + zk \) be the vector drawn from the origin to the point \( P = (x,y,z) \). Suppose that we move from \( P \) to a nearby point \( Q = (x+\Delta x,y+\Delta y,z+\Delta z) \).

Then \( \Phi \) will change by an amount \( \Delta \Phi \) where

\[
\Delta \Phi = \Phi_x \Delta x + \Phi_y \Delta y + \Phi_z \Delta z + \epsilon_1 \Delta x + \epsilon_2 \Delta y + \epsilon_3 \Delta z
\]

where \( \epsilon_1, \epsilon_2, \) and \( \epsilon_3 \to 0 \) as the point \( Q \to P \). If we divide the change \( \Delta \Phi \) by the distance \( \Delta s = |\Delta \vec{r}| \) between \( P \) and \( Q \), we obtain a measure of the rate at which \( \Phi \) changes when we move from \( P \) to \( Q \):

\[
\frac{\Delta \Phi}{\Delta s} = \Phi_x \frac{\Delta x}{\Delta s} + \Phi_y \frac{\Delta y}{\Delta s} + \Phi_z \frac{\Delta z}{\Delta s} + \epsilon_1 \frac{\Delta x}{\Delta s} + \epsilon_2 \frac{\Delta y}{\Delta s} + \epsilon_3 \frac{\Delta z}{\Delta s}
\]

Example:

If \( \Phi(x,y,z) \) represents the temperature at any point \( P(x,y,z) \) then \( \frac{\partial \Phi}{\partial s} \) is the average rate of change in temperature per unit length at the point \( P \) in the direction in which \( \Delta s \) is measured.

The limiting value of \( \frac{\Delta \Phi}{\Delta s} \) as \( \Delta s \to 0 \), that is, as \( Q \to P \) along the segment \( PQ \), is called the derivative of \( \Phi \) in the direction \( PQ \) or simply the directional derivative of \( \Phi \). Since \( \epsilon_1, \epsilon_2, \epsilon_3 \to 0 \) as \( Q \to P \), we have that

\[
\frac{d\Phi}{ds} = \frac{\partial \Phi}{\partial x} \frac{dx}{ds} + \frac{\partial \Phi}{\partial y} \frac{dy}{ds} + \frac{\partial \Phi}{\partial z} \frac{dz}{ds}
\]

The first factor in each term of the products in the expression above for the directional derivative depend only on \( \Phi \) and the point \( P \). The second factors in the products are independent of \( \Phi \) and depend on the direction in which the derivative is being computed. We may rewrite the expression above in the form
\[
\frac{d\Phi}{ds} = (\Phi_x \vec{i} + \Phi_y \vec{j} + \Phi_z \vec{k}) \cdot \left( \frac{dx}{ds} \vec{i} + \frac{dy}{ds} \vec{j} + \frac{dz}{ds} \vec{k} \right) = (\Phi_x \vec{i} + \Phi_y \vec{j} + \Phi_z \vec{k}) \cdot \frac{d\vec{r}}{ds}
\]

The vector \( \Phi_x \vec{i} + \Phi_y \vec{j} + \Phi_z \vec{k} \) is known as the gradient of \( \Phi \) or \( \text{grad} \Phi \). Thus

\[\text{grad} \Phi = \Phi_x \vec{i} + \Phi_y \vec{j} + \Phi_z \vec{k}\]

The notation \( \nabla \Phi \) is often used for \( \text{grad} \Phi \). In this notation the operator \( \nabla \) is defined as

\[
\nabla = \vec{i} \frac{\partial}{\partial x} + \vec{j} \frac{\partial}{\partial y} + \vec{k} \frac{\partial}{\partial z}
\]

**Example:**
Let \( \Phi(x,y,z) = xyz + 3x^4y^2z^3 \). Then \( \nabla \Phi = (yz + 12x^3y^2z^3)\vec{i} + (xz + 6x^4yz^3)\vec{j} + (xy + 9x^4y^2z^2)\vec{k} \)

**Example:**
We may use SNB to find the gradient of a function. However, SNB writes vectors as ordered triples instead of the form given in the previous example. Thus

\[
\nabla(xyz + 3x^4y^2z^3) = (yz + 12x^3y^2z^3, xz + 6x^4yz^3, xy + 9x^4y^2z^2)
\]

With this notation we may write the directional derivative of \( \Phi \) in the form

\[
\frac{d\Phi}{ds} = \text{grad} \Phi \cdot \frac{d\vec{r}}{ds} = \nabla \Phi \cdot \frac{d\vec{r}}{ds}
\]

Remark: Since \( \Delta s \) is the length of \( \Delta \vec{r} \) then \( \frac{\Delta \vec{r}}{\Delta s} \) and hence \( \frac{d\vec{r}}{ds} \) are unit vectors. Therefore, \( \nabla \Phi \cdot \frac{d\vec{r}}{ds} \) is the projection of \( \text{grad} \Phi \) in the direction of \( \frac{d\vec{r}}{ds} \). Thus \( \nabla \Phi \) has the property that its projection in any direction is equal to the derivative of \( \Phi \) in that direction. Since the maximum projection of a vector is the vector itself, it is clear that \( \text{grad} \Phi \) extends in the direction of the greatest rate of change of \( \Phi \) and has that rate of change for its length.

**Example:**
What is the directional derivative of the function \( \Phi = xy^2 + yz^3 \) at \( (2, -1, 1) \) in the direction of the vector \( \vec{i} + 2\vec{j} + k \)?

\[
\nabla(xy^2 + yz^3) = (y^2, 2xy + z^3, 3yz^2)
\]

and a unit vector in the given direction is \( \frac{1}{3}(1, 2, 2) \). Thus
\[
\frac{d\Phi}{ds} = (y^2, 2xy + z^3, 3yz^2) \cdot \frac{1}{3}(1, 2, 2) = \frac{1}{3}y^2 + \frac{4}{3}xy + \frac{2}{3}z^3 + 2yz^2
\]

Hence

\[
\left.\frac{d\Phi}{ds}\right|_{(2,-1,1)} = \frac{1}{3}(-1)^2 + \frac{4}{3}(2)(-1) + \frac{2}{3}(1)^3 + 2(-1)(1)^2 = -\frac{11}{3}.
\]

Remark There is a very nice discussion of the gradient at Gradient. There is a discussion of the gradient as well as a couple of very nice Java applets. This site was done at RPI.

Let us now consider the operator

\[
\nabla = i \frac{\partial}{\partial x} + j \frac{\partial}{\partial y} + k \frac{\partial}{\partial z}.
\]

Given any other vector \( \vec{F} = F_1 i + F_2 j + F_3 k \) where \( \vec{F} = \vec{F}(x,y,z) \) we can consider \( \nabla \cdot \vec{F} \), called the divergence of \( \vec{F} \), and \( \nabla \times \vec{F} \), called the curl \( \vec{F} \).

\[
\nabla \cdot \vec{F} = \left( i \frac{\partial}{\partial x} + j \frac{\partial}{\partial y} + k \frac{\partial}{\partial z} \right) \cdot \left( F_1 i + F_2 j + F_3 k \right) = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = \text{div}\vec{F}
\]

\[
\nabla \times \vec{F} = \text{curl}\vec{F} = \left( \frac{\partial}{\partial x} \frac{\partial}{\partial z} - \frac{\partial}{\partial z} \frac{\partial}{\partial y} \right) i + \left( \frac{\partial}{\partial z} \frac{\partial}{\partial x} - \frac{\partial}{\partial x} \frac{\partial}{\partial z} \right) j + \left( \frac{\partial}{\partial y} \frac{\partial}{\partial x} - \frac{\partial}{\partial x} \frac{\partial}{\partial y} \right) k
\]

\[
= \left| \begin{array}{ccc}
\vec{i} & \vec{j} & \vec{k} \\
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
F_1 & F_2 & F_3
\end{array} \right|
\]

Example Let \( \vec{F} = 2x\vec{i} + 3y^2\vec{j} + 2xz\vec{k} \). Find \( \nabla \cdot \vec{F} \) and \( \nabla \times \vec{F} \).

\[
\text{div}\vec{F} = 2 + 6y + 2z
\]

\[
\nabla \times \vec{F} = \left| \begin{array}{ccc}
\vec{i} & \vec{j} & \vec{k} \\
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
2x & 3y^2 & 2xz
\end{array} \right| = \vec{i}(0 + 0) - \vec{j}(0 - 0) + \vec{k}(0 - 2j)
\]