Ma 227 Homework 10 Solutions Fall 2009
Due 11/19/2009

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1) Evaluate the line integral by two methods: a) directly, and b) using Green’s Theorem.
\[ \oint_C xy^2 dx + x^3 dy \]
C is the rectangle with vertices (0, 0)(2, 0)(2, 3)(0, 3)

a) directly: Let \( C_1 \) be the segment from (0, 0) to (2, 0), \( C_2 \) the segment from (2, 0) to (2, 3), \( C_3 \) the segment from (2, 3) to (0, 3), \( C_4 \) the segment from (0, 3) to (0, 0)

\( C_1 \to x = t, dx = dt; y = 0, dy = 0 dt \quad 0 \leq t \leq 2 \)
\( C_2 \to x = 2, dx = 0 dt; \quad y = t, dy = dt \quad 0 \leq t \leq 3 \)
\( C_3 \to x = 2 - t, dx = -dt; \quad y = 3, dy = 0 \quad 0 \leq t \leq 2 \)
\( C_4 \to x = 0, dx = 0 dt; \quad y = 3 - t, dy = -dt \quad 0 \leq t \leq 3 \)

Thus \( \oint_C xy dx + x^2 y^3 dy = \oint_{C_1+C_2+C_3+C_4} (xy dx + x^2 y^3 dy) \)
\[ = \int_0^2 0 dt + \int_0^3 8 dt + \int_0^2 -9(2 - t) dt + \int_0^3 0 dt = 0 + 24 - 18 + 0 = 6 \]

b) Using Green’s Theorem:
\[ \oint_C xy^2 dx + x^3 dy = \iint_D \left[ \frac{\partial}{\partial x} (x^3) - \frac{\partial}{\partial y} (xy^2) \right] dA = \int_0^2 \int_0^3 (3x^2 + 2xy) dy dx = \int_0^2 (9x^2 - 9x) = 24 - 3 \]

3) Evaluate the line integral by two methods: a) directly, and b) using Green’s Theorem.
\[ \oint_C xy dx + x^2 y^3 dy \]
(0, 0, 1, 0, 1, 2, 0, 0)
a) directly: Let $C_1$ be the segment from $(0,0)$ to $(2,0)$, $C_2$ the segment from $(2,0)$ to $(2,3)$, $C_3$ the segment from $(2,3)$ to $(0,3)$, $C_4$ the segment from $(0,3)$ to $(0,0)$

**a) directly:** Let $C_1$ be the segment from $(0,0)$ to $(1,0)$, $C_2$ the segment from $(1,0)$ to $(1,2)$ and $C_3$ the segment from $(1,2)$ to $(0,0)$.

$C_1 \rightarrow x = t, dx = dt, y = 0, dy = 0 \ 0 \leq t \leq 1$

$C_2 \rightarrow x = 1, dx = 0 \ dt; \ y = t, dy = dt \ 0 \leq t \leq 2$

$C_3 \rightarrow \text{The equation of this line is } y = 2x. \ Thus x = t, dx = dt; y = 2t, dy = 2dt \ t : 1 \rightarrow 0$

$$\int_C xydx + x^2y^3dy = \int_{C_1} xydx + x^2y^3dy + \int_{C_2} xydx + x^2y^3dy + \int_{C_3} xydx + x^2y^3dy$$

$$= \int_0^1 0 dt + \int_0^1 t^{3} dt + \int_0^1 t(2t) dt + \int_0^1 t^2(8t^3)(2dt) = \int_0^1 t^{3} dt + \int_0^1 (2t^2) dt + 2 \int_0^1 (8t^5) dt = \frac{2}{3}$$

b) Using Greens Theorem:

$$\oint_C xydx + x^2y^3dy = \iint_D \left( \frac{\partial}{\partial x} (x^2y^3) - \frac{\partial}{\partial y} (xy) \right) dA$$

$$= \int_0^1 \int_0^2 (2xy^3 - x) dy dx$$

$$= \frac{2}{3}$$

5)

$P(x,y) = x^4y^5; \ Q(x,y) = -x^7y^6$

$C \rightarrow x^2 + y^2 = 1$

Since $C$ is a circle of radius 1, we parametrize $C$ as

$x = \cos \theta, dx = -\sin \theta d\theta \ \ 0 \leq \theta \leq 2\pi$

$y = \sin \theta, dy = \cos \theta d\theta$

$$\int_C x^4y^5dx - x^7y^6dy = \int_0^{2\pi} \cos^4 \theta \sin^5 \theta (-\sin \theta) d\theta - \int_0^{2\pi} \cos^7 \theta \sin^6 \theta (\cos \theta) d\theta$$

$$= -\int_0^{2\pi} (\cos^4 \theta \sin^6 \theta + \cos^8 \theta \sin^6 \theta) d\theta$$

$$= -\frac{29}{1024}\pi$$

Note that it was necessary to simply the integral in order to have SNB evaluate it.

Double Integral:
\[ x^2 + y^2 = 1 \]
x goes from \(-1\) to 1  
y goes from \(-\sqrt{1-x^2}\) to \(\sqrt{1-x^2}\)

\[ \int_D [\partial/\partial x (Q) - \partial/\partial y (P)] dA = \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (-7x^6y^6 - 5x^4y^4) \, dy \, dx = -\frac{29}{1024} \pi \]

7) Use Green’s Theorem to evaluate the line integral along the given positively oriented curve.
\[ \int_C e^y \, dx + 2xe^y \, dy, \ C \text{ is the square with sides } x = 0, x = 1, y = 0 \text{ and } y = 1 \]
\[ (0,0,0,1,1,1,1,0) \]

The region \(D\) enclosed by \(C\) is \([0, 1] \times [0, 1]\), so
\[ \int_C e^y \, dx + 2xe^y \, dy = \int_D [\frac{\partial}{\partial x} (2xe^y) - \frac{\partial}{\partial y} (e^y)] \, dA = \int_0^1 \int_0^1 (2e^y - e^y) \, dy \, dx = \int_0^1 \int_0^1 \left(2 - 1\right) \, dy \, dx = e - 1 \]

13) \(F(x,y) = \langle \sqrt{x} + y^3, x^2 + \sqrt{y} \rangle, \)
\(C\) consists of the arc of the curve \(y = \sin x\) from \((0, 0)\) to \((\pi, 0)\) and the line segment from \((\pi, 0)\) to \((0, 0)\)
\(C\) is traversed clockwise, so \(-C\) gives the positive orientation
\[ \int_C F \cdot dr = -\int_{-C} ((\sqrt{x} + y^3) \, dx + (x^2 + \sqrt{y}) \, dy = -\int_D \left[ \frac{\partial}{\partial x} (\sqrt{x} + y^3) - \frac{\partial}{\partial y} (x^2 + \sqrt{y}) \right] \, dA \]
\[ = -\int_0^\pi \int_0^{\sin x} (2x - 3y^2) \, dy \, dx = -\int_0^\pi [2xy - y^3]_{y=0}^{y=\sin x} \, dx \]
\[ = -\int_0^\pi (2x \sin x - \sin^3 x) \, dx = -\left[ \int_0^\pi (2x \sin x - (1 - \cos^2 x) \sin x) \, dx \right] \]
\[ = -\left[ \int_0^\pi 2 \sin x - 2x \cos x + \cos x - \frac{1}{3} \cos^3 x \right]_0^\pi = -\left[ \left(2\pi - 2 + \frac{2}{3}\right) - \left(\frac{2}{3}\right) \right] = \frac{4}{3} - 2\pi \]

15) \(F(x,y) = \langle e^x + x^2 y, e^y - xy^2 \rangle, \)
\(C\) is the circle \(x^2 + y^2 = 25\) orientated clockwise.
\(C\) is traversed clockwise, so \(-C\) gives the positive orientation
\[ \int_C F \cdot dr = -\int_{-C} (e^x + x^2 y) \, dx + (e^y - xy^2) \, dy = -\int_D \left[ \frac{\partial}{\partial x} (e^y - xy^2) - \frac{\partial}{\partial y} (e^x + x^2 y) \right] \, dA \]
\[ = -\int_D (-y^2 - x^2) \, dA = \int_D (x^2 + y^2) \, dA = \int_0^{2\pi} \int_0^5 (r^2) r \, dr \, d\theta \]
17) Use Green’s Theorem to find the work done by the force $F(x, y) = x(x + y)i + xy^2j$ in moving a particle from the origin along the x-axis to (1, 0) then along the line segment to (0, 1), and the back to the origin along the y-axis. The path is shown below.

(0, 0, 1, 0, 0, 1, 0, 0)

The line joining (1, 0) to (0, 1) has equation $y = 1 - x$. Thus, by Greens Theorem,

$$\text{Work} = \int_C F \cdot dr = \iint_D (y^2 - x)dydx$$

where $C$ is the path described in the question and $D$ is the triangle bounded by $C$.

$$\text{Work} = \int_0^1 \int_0^{1-x} (y^2 - x)dydx = -\frac{1}{12}$$

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5) Evaluate $\iiint_S yz dS$ for $S: x = uv$, $y = u + v$, $z = u - v$, $u^2 + v^2 \leq 1$

$$\iiint_S yz dS = \iiint_D f(r(u, v)) |r_u \times r_v| dA$$

define: $\vec{r}(u, v) = (uv, (u + v), (u - v)) k$

$$\vec{r}_u = \left( \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u} \right) = (v, 1, 1)$$

$$\vec{r}_v = \left( \frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v} \right) = (u, 1, -1)$$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} i & j & k \\ v & 1 & 1 \\ u & 1 & -1 \end{vmatrix} = -2i + (u + v)j + (v - u)k$$

$$|\vec{r}_u \times \vec{r}_v| = \sqrt{2^2 + (u + v)^2 + (u - v)^2} = \sqrt{4 + 2u^2 + 2v^2}$$
\[ \int_s yz dS = \int_d x^2 yz \sqrt{\left( \frac{\partial z}{\partial x} \right)^2 + \left( \frac{\partial z}{\partial y} \right)^2 + 1} \, dA \]

\[ \int_0^3 \int_0^{(1 - x - y)} \sqrt{(-1)^2 + (-1)^2 + 1^2} \, dA \]

\[ \int_0^1 (\sqrt{3} y(1 - x - y)) dy dx = \frac{1}{24} \sqrt{3} \]

15) \[ \int_s (x^2 z + y^2 z) dS \]

\[ S \text{ is the hemisphere } x^2 + y^2 + z^2 = 4, \ z \geq 0 \]

Using spherical coordinates:
\[ \vec{r}(\phi, \theta) = 2 \sin \phi \cos \theta \vec{i} + 2 \sin \phi \sin \theta \vec{j} + 2 \cos \phi \vec{k} \]
\[ ||\vec{r}_\theta \times \vec{r}_\phi|| = 4 \sin \phi \]
\[ \int_s (x^2 z + y^2 z) dS = \int_0^{2\pi} \int_0^{\frac{\pi}{2}} (4 \sin^2 \phi)(2 \cos \phi)(4 \sin \phi) d\phi d\theta = [16\pi \sin^4 \phi]_0^{\frac{\pi}{2}} = 16\pi \]