GENERAL DEVIATION MEASURES
AND PORTFOLIO ANALYSIS

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Outline

- Introduction
- General Deviation Measures
- Portfolio Optimization
- Capital Asset Pricing Model
- Concluding Remarks
Standard Deviation

\[ \sigma(X) = \sqrt{E[X - EX]^2} \]

Properties

(D1) – insensitivity to constant shift
\[ \sigma(X + C') = \sigma(X) \text{ for all } X \text{ and constants } C' \]

(D2) – positive homogeneity
\[ \sigma(\lambda X) = \lambda \sigma(X) \text{ for all } X \text{ and all } \lambda > 0 \]

(D3) – subadditivity
\[ \sigma(X + X') \leq \sigma(X) + \sigma(X') \text{ for all } X \text{ and } X' \]

(D4) – nonnegativity
\[ \sigma(X) \geq 0 \text{ (equality for constant } X) \]
Deviation Measures

Motivation

- Standard deviation
- Risk preferences
- Markowitz’s model
- CAPM

General deviation measures

- Axioms including standard deviation
- Nonsymmetric measures
- Non-differentiability

Investors

$\sigma(X)$

$D(X)$

Optimal investment
System of Axioms

- (D1) – insensitivity to constant shift
  \[ D(X + C) = D(X) \text{ for all } X \text{ and constants } C \]

- (D2) – positive homogeneity
  \[ D(\lambda X) = \lambda D(X) \text{ for all } X \text{ and all } \lambda > 0 \]

- (D3) – subadditivity
  \[ D(X + X') \leq D(X) + D(X') \text{ for all } X \text{ and } X' \]

- (D4) – nonnegativity
  \[ D(X) \geq 0 \text{ (equality for constant } X) \]

- (D5) – coherency
  \[ D(X) \leq EX - \inf X \text{ for all } X \]
Examples of Deviation Measures

- **Standard Deviation**

\[ \sigma(X) = \left( E[X - EX]^2 \right)^{1/2} \]

- **Standard Semideviations**

\[ \sigma_+(X) = \left( E[\max\{X - EX, 0\}^2] \right)^{1/2} \]
\[ \sigma_-(X) = \left( E[\max\{EX - X, 0\}^2] \right)^{1/2} \]

- **Deviation measure from range**

\[ EX - \inf X \]

- **Generalized Mean Absolute Deviation (GMAD) with** \[ a(\omega) > 0 \]

\[ \text{MAD}(X) = \int_{\Omega} a(\omega)|X(\omega) - EX|dP(\omega) \]
**Deviation CVaR**

Deviation Conditional Value-at-Risk (CVaR) for $\alpha \in [0, 1]$

\[
\text{CVaR}_\alpha^\Delta (X) = EX - \frac{1}{\alpha} \int_0^\alpha (-\text{VaR}_p(X)) dp
\]
Mixed Deviation CVaR

\[ \lambda_k \geq 0 \text{ and } \sum_{k=1}^{K} \lambda_k = 1 \]

\[ \text{Mixed-CVaR}_{\Delta}(X) = \sum_{k=1}^{K} \lambda_k \text{CVaR}_{\alpha_k}(X) \]

\[ \lambda(\alpha) > 0 \text{ and } \int_0^1 \lambda(\alpha) d\alpha = 1 \]

\[ \text{Mixed-CVaR}_{\alpha}(X) = \int_0^1 \text{CVaR}_{\alpha}(X) d\lambda(\alpha) \]
Convex Analysis

**Subgradient (generalization of derivative)**

$G \in \mathcal{L}^2$ is a **subgradient** of $\mathcal{D}$ at $X$ if

$$\mathcal{D}(X') \geq \mathcal{D}(X) + E[G(X' - X)] \quad \text{for all} \quad X' \in \mathcal{L}^2$$

$$\partial \mathcal{D}(X) = \text{set of all such } G \text{ at } X$$

**Risk envelope (duality)**

- $Q = \{ Q \in \mathcal{L}^2 \mid E[(1 - Q)X] \leq \mathcal{D}(X) \text{ for all } X, \ E Q = 1 \}$

- $\mathcal{D}(X) = \sup_{Q \in Q} \text{covar}(Q, -X)$

$\mathcal{D}$ is coherent $\iff Q \geq 0$
Portfolio Optimization

Market
- risk-free asset with constant rate $r_0$
- $n$ risky assets with rates $r = (r_1, \ldots, r_n)$, $r_j \in \mathcal{L}^2$

Portfolio
- weights: $x_0, x = (x_1, \ldots, x_n)$
- budget constraint: $x_0 + x^\top e = 1$
- rate of return: $X_p = x_0r_0 + x^\top r$

No Redundancy Assumption
Only 0-portfolio, has risk-free return

$$
D(x^\top r) > 0 \text{ for all } x \neq 0
$$

No arbitrage Assumption
No $x$-portfolio has a risk-free return $> r_0$
Problem Formulation

Risk preferences

Investor 1

\[ D_1(X) = \sigma(X) \]

Investor 2

\[ D_2(X) \]

Investor 3

\[ D_3(X) \]

Problem formulation

\[
\begin{align*}
    d_0(\Delta) &= \min_{x_0, x} \mathcal{D}(X_p) \\
    \text{s. t.} & \quad EX_p = r_0 + \Delta \\
    & \quad x_0 + x^\top e = 1
\end{align*}
\]

\[ \Delta = \text{demanded additional gain over the risk-free rate} \]
Portfolio Decomposition

Portfolio Decomposition

\[ X_p = x_0 r_0 + (1-x_0) Y_p \]

- Risk free rate \( r_0 \)
- Portfolio of risky assets \( Y_p = r_0 y_1 + \ldots + r_n y_n \)

- low \( r_0 \) \( \Rightarrow y_1 + \ldots + y_n = 1 \)
- high \( r_0 \) \( \Rightarrow y_1 + \ldots + y_n = -1 \)
- some \( r_0 \) \( \Rightarrow y_1 + \ldots + y_n = 0 \)
Efficient Set: Classical Theory

One Fund Theorem

Auxiliary Problem

\[
\min_{Y} \sigma(Y_p) \\
\text{s. t. } EY_p = \zeta \\
y^\top e = 1
\]
Threshold Value: Standard Deviation

Master fund of negative type

\[-\sigma(-1,-\zeta)\]

\[\sigma(1,\zeta)\]

\[\sigma(-1,\zeta) = \sigma(1,-\zeta)\]

\[\hat{r}_0\]

\[r_0 + \Delta\]
Efficient Sets: General Deviation

- \( d(-1, -\zeta) \)
- \( d(1, \zeta) \)
- Master fund of negative type
- Master fund of positive type
- \( r_0 \)
- \( r_0^+ \)
- \( r_0^- \)
- convex slopes
- non-smooth
- non-symmetric
- threshold interval
Threshold Values

Auxiliary Problem: zero-price risky portfolio

\[
\min_y \mathcal{D}(Y_p) \\
\text{s. t.} \quad EY_p = 1 \\
y^\top e = 0
\]

Lagrange relaxation

\[
L(y, \rho, \eta) = \mathcal{D}(y^\top r) - \rho y^\top e + \eta(1 - y^\top \bar{r})
\]

Proposition (threshold set of risk-free rates)

\[
\hat{r}_0 = -\frac{\rho^*}{\eta^*}
\]

\[
\rho^* \in [\rho^-, \rho^+] \quad \text{and} \quad \eta^* = \mathcal{D}(y^{*\top} r)
\]
\textbf{Reduced Optimization Perspective}

\begin{align*}
\text{Auxiliary Problem} \\
\min_x & \quad \mathcal{D}(x^\top r) \\
\text{s. t.} & \quad x^\top \bar{r} = \zeta \\
& \quad x^\top e = \pi
\end{align*}

\textbf{Reformulation}

\begin{align*}
\quad d(\pi, \zeta) \\
\text{s. t.} & \quad \zeta = r_0 \pi + 1
\end{align*}

\textbf{Threshold for } r_0 \\
\text{Non-smooth } d(0, 1)
Generalized One Fund Theorem

Existence of Master Funds

The threshold values \( \hat{r}^+_0 \) and \( \hat{r}^-_0 \) have the property that

- \( r_0 < \hat{r}^-_0 \implies \) master fund of positive type
  but none of negative type

- \( r_0 > \hat{r}^+_0 \implies \) master fund of negative type
  but none of positive type

- \( \hat{r}^-_0 < r_0 < \hat{r}^+_0 \implies \) neither master fund of positive type
  nor one of negative type
Capital Asset Pricing Model (CAPM)

Assumption: low risk-free rate, \( r_0 < r_0^- \)

- **Optimization problem**

\[
\begin{align*}
\min_x & \quad \mathcal{D}(x^\top r) \\
\text{s. t.} & \quad x^\top (\bar{r} - r_0 e) = \Delta
\end{align*}
\]

Master fund of positive type: \( e^\top x^*(\Delta) = 1, \; r_M = x^*\top r \)

- **Optimality conditions (master fund)**

\[
\frac{\mathcal{D}(r_M)}{\bar{r}_M - r_0} (\bar{r}_i - r_0) = E[r_i G], \quad G \in \partial \mathcal{D}(r_M)
\]

- **Capital Asset Pricing Model (CAPM)**

\[
\bar{r}_i - r_0 = \beta_i (\bar{r}_M - r_0), \quad \beta_i = \frac{\text{covar}(G, r_i)}{\mathcal{D}(r_M)}
\]
Market Equilibrium

Classical covariance relations: one-factor predictive model (CAPM)

\[ r_i - r_0 \approx \beta_i [r_M - r_0] \quad \text{for } i = 1, \ldots, n. \]

- \( \beta_i \) may not be uniquely determined
- \( r_M \) nonuniqueness: “flat spot” on the efficient frontier
**CAPM: Standard Deviation**

- **Subgradient set:** \( G \in \partial D(r_M) \)
  \[
  G = \frac{r_M - \bar{r}_M}{\sigma(r_M)}
  \]

- **Asset’s \( \beta_i \)**
  \[
  \beta_i = \frac{\text{covar}(r_i, r_M)}{\sigma^2(r_M)}
  \]

- **Threshold value for \( r_0 \)**
  \[
  \hat{r}_0 = \frac{\bar{r}^\top \Lambda^{-1} e}{\bar{e}^\top \Lambda^{-1} e}
  \]

\( \Lambda \) is variance-covariance matrix
**CAPM-like Relations: Semideviation**

**Lower semideviation**

\[
\sigma_-(X) = (E[\max\{EX - X, 0\}^2])^{1/2}
\]

- Subgradient set: \( G \in \partial \mathcal{D}(r_M) \)

\[
G = \frac{r_M^- - Er_M^-}{\sigma_-(r_M)}
\]

where \( r_M^- = \min\{r_M - Er_M, 0\} \)

- **CAPM-like relation**

\[
\beta_i = \frac{\text{covar}(r_i, r_M^-)}{\sigma^2(r_M)}
\]
CAPM-like Relations: GMAD

Generalized Mean Absolute Deviation

\[ D(X) = E\left[ a(\omega)|X(\omega) - E X| \right] \]

- subgradient set \( G^* = 1 - Q^* \in \partial D(r_M), G^* = V - EV \)

\[
V(\omega) \begin{cases} 
  = a(\omega) & \text{on } \{ \omega \in \Omega \mid r_M(\omega) > \bar{r}_M \} \\
  = -a(\omega) & \text{on } \{ \omega \in \Omega \mid r_M(\omega) < \bar{r}_M \} \\
  \in [-a(\omega), a(\omega)] & \text{on } \{ \omega \in \Omega \mid r_M(\omega) = \bar{r}_M \}
\end{cases}
\]

- If \( r = (r_1, \ldots, r_n) \) continuously distributed, then

\[
\beta_i = \frac{E \left[ a(r_i - \bar{r_i}) \text{sign}(r_M - \bar{r}_M) \right]}{E\left[ a|r_M - \bar{r}_M| \right]}
\]
CAPM-like Relations: Deviation CVaR

Deviation Conditional Value-at-Risk

\[ CVaR_{\alpha}^\Delta (X) = E X - \frac{1}{\alpha} \int_{0}^{\alpha} (-VaR_p(X)) \, dp \]

- Subgradient set \( G \in \partial D(r_M) \), \( EG = 0 \)

\[
G(\omega) \begin{cases} 
= 1 - \alpha^{-1} & \text{on } \{ \omega \mid r_M(\omega) < -VaR_\alpha(r_M) \} \\
\in [1 - \alpha^{-1}, 1] & \text{on } \{ \omega \mid r_M(\omega) = -VaR_\alpha(r_M) \} \\
= 1 & \text{on } \{ \omega \mid r_M(\omega) > -VaR_\alpha(r_M) \}
\end{cases}
\]

- If \( r = (r_1, \ldots, r_n) \) continuously distributed, then

\[
\beta_i = \frac{E \left[ r_i - \bar{r}_i \mid r_M(\omega) \leq F_{r_M}^{-1}(\alpha) \right]}{CVaR_{\alpha}^\Delta (r_M)}
\]
### Concluding Remarks

Developed rigorous optimization framework for portfolio analysis with general deviation measures (non-differentiability, discrete and continuous distributions)

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