Ma 635. Real Analysis I. Hw5, due 10/05.

HW 5. (due 10/05). Solutions.

1. [1] p. 92 # 9
Give an example of a closed bounded subset of $l_\infty$ that is not totally bounded.

Solution. Just the set of unit coordinate vectors like $(0, 0, 1, 0, 0, \ldots)$.

2. [1] p. 110 # 12
Show that the set $A = \{x \in l_2 : |x_n| \leq \frac{1}{n}\}$ is compact in $l_2$.

Done in class, compare with the next problem.

3. [1] p. 110 # 14
Show that the Hilbert cube $H^\infty = \{x = (x_n)_{n=1}^\infty, |x_n| \leq 1\}$ is compact if

$$d(x, y) = \sum_{n=1}^\infty \frac{|x_n - y_n|}{2^n}.$$ 

Solution. \(\forall \varepsilon\) we first choose $N$ such that \(\sum_{n=N+1}^\infty \frac{1}{2^n} < \varepsilon/2\) (for example, \(N > \log_2 \frac{1}{\varepsilon}\)). Then for every coordinate from 1 to $N$ we choose $2N+1$ grid points \(\frac{k}{2N}\) that are distant by \(\varepsilon/(2N)\) from each other, \(-N \leq k \leq N\). To form the \(\varepsilon\)-net, we pick up all possible grid points for first $N$ coordinates, and zeros for the remaining infinite "tail".

4. [1] p. 110 # 17
If $M$ is compact, show that $M$ is also separable.

Solution. In view of total boundedness, for any $\varepsilon_1 = 1, \frac{1}{2}, \frac{1}{3}, \ldots \frac{1}{n}, \ldots$ the exists a bounded \(\varepsilon_n\)-net. The union of these \(\varepsilon_n\)-nets is just a dense countable subset.

5. [1] p. 110 # 18
$M$ has a countable open base if and only if $M$ is separable.

Solution. \((\Rightarrow)\) If $M$ has a countable open base, we choose a point inside each element of the base. These points form a dense and separable subset.

\((\Leftarrow)\) If $M$ is separable then consider the collection of open balls with rational radiuses centered at the elements of the dense countable subset of $M$.

If $M$ is compact and $f : M \to N$ is a continuous bijection then $M$ is homeomorphism.

Solution. We need to show only that $f^{-1}$ is also continuous. Let $y_n \to y$ in $N$. Consider $x_n$ - their pre-images in $M$, $x_n = f^{-1}(y_n), x = f^{-1}(y)$.

Since $M$ is compact then sequence $\{x_n\}$ contains a converging subsequence $\{x_{n_k}\}$ with limit, say, $x$. Since $f$ is continuous then $f(x_{n_k}) \to f(x)$, or $f(y_{n_k}) \to f(z)$. However, since $y_n \to y$ then also $y_{n_k} \to y$ and, hence, $f(z) = y$. Then $z = f^{-1}(y) = x$. Consequently, $x_{n_k} \to x$. Consider the terms of $\{x_n\}$ outside the subsequence $\{x_{n_k}\}$. Because of the above argument, those remaining terms cannot have a subsequence that converges to a value other than $x$. Since all converging subsequences of $\{x_n\}$ converge to $x$, then $x_n \to x$, which proves the continuity of $f^{-1}$.

7. [1] p. 114 # 34
$A$ is closed in $M$ \iff $A \cap K$ is compact $\forall K$-compact.

Solution. \((\Rightarrow)\) As a subset of $K$, $A \cap K$ is totally bounded. As an intersection of two closed sets, $A \cap K$ is closed. Then it is compact.

\((\Leftarrow)\) We consider a Cauchy sequence $\{x_n\} \subset A$. Let $K = \{x_n\} \cup x$ where $x$ is the limit of $x_n$. Since $A \cap K$ is compact then $x \in A$.

Every open cover $\mathcal{G}$ of compactum $M$ has a Lebesgue number $L(\mathcal{G}) > 0$. By definition,

$$L(\mathcal{G}) = \inf_{\varepsilon > 0} \{ \forall B_\varepsilon \subset M \ \exists G \in \mathcal{G} : B_\varepsilon \subseteq G \}.$$  

**Solution.** If a cover $\mathcal{G}$ has no positive Lebesgue number, then it is equal to zero. Then $\forall \{\varepsilon_n\}_{n=1}^\infty, \varepsilon_n \to 0, \exists B_{\varepsilon_n} \subset M$ such that $\forall G \in \mathcal{G}, B_{\varepsilon_n} \not\subseteq G$. Let $\{G_n\}_{n=1}^N$ be a finite open subcover of $\mathcal{G}$ that covers $M$. We consider the open subsets $D_n = B_{\varepsilon_n} \setminus G_n$. They are non-empty (otherwise $B_{\varepsilon_n} \subset G_n$). Their intersection is also non-empty (why?). Then $\bigcap_{n=1}^N D_n \not\subseteq M$, which contradicts the inclusion $D_n \subseteq M$ for all $n$.

9 [1] p. 114 # 36

10 [2], p. 81, # 4. Show that $E = \{1/n : n \text{ a positive integer}\}$ is not compact in $\mathbb{R}$ but $E \cup \{0\}$ is compact.

11 [2], p. 81, # 7. Prove that every finite subset of a metric space is compact.

**Solution.** Any sequence from a finite subset contains a stationary subsequence, which is convergent.

12 [2], p. 81, # 6. Show that a discrete metric space $M$ is not compact unless $X$ is finite.

**Solution.** Let $x_n$ be a countable infinite sequence in $M$. Since the distance between any two points is equal to 1, this sequence has no converging subsequence and, thus, is not compact.

13 [2], p. 81, # 9

If $X$ is compact prove that $C(X, \mathbb{R})$ is a complete metric space.

Slightly modify the theorem from Advanced Calculus that states that the uniform convergence of continuous functions yield a continuous function as the pointwise limit.

14 [2], p. 81, # 10

Is $C[0, 1]$ compact?

No, $x_n(t) = t^n$ is a not relatively compact subset, its pointwise limit is a discontinuous function.

15. [2], p. 84, # 4 Prove that any compact metric space has a dense countable subset.

See # 4.

16. [3], p. 115, # 5

Let $X$ be a metric compactum and $A : X \mapsto X$ such that $d(Ax, Ay) < d(x, y)$ if $x \neq y$. Prove that $A$ has a unique fixed point.

**Solution.** Consider arbitrary $x_0$ and let $x_n = Ax_0$. Let $d_1 = d(x_0, x_1)$, $d_2 = d(x_1, x_2)$. ... As we know, $d_1 > d_2 > d_3 > \cdots$. Since $X$ is compact, we pick up a converging subsequence of $\{x_n\}$, $x_n \to z_0$ as $k \to \infty$. That implies $d(x_{n_k}, x_{m_k}) \to 0$ as $k, m \to \infty$. We consider the sequence $Ax_{n_k} = x_{n_k+1}$. From the problem statement, $d(Ax_{n_k}, Ax_{m_k}) < d(x_{n_k}, x_{m_k}) \to 0$. Hence, the sequence $Ax_{n_k+1}$ is also a Cauchy sequence, let its limit be $z_1$. Similarly, we obtain $\exists \lim x_{n_k+2} = z_2, \exists \lim x_{n_k+3} = z_3$, and so on. The number of limit points $z_0, z_1, z_2, \ldots$ cannot exceed $n_k$ since the sequence $x_{n_k+1}$ is just $x_{n_k}$ and, hence, $x_{n_k+1} \to z_0$. So, we have limit points $z_0, z_1, \ldots, z_p$. The condition $d(Ax, Ay) < d(x, y)$ implies that $A$ is a continuous mapping. Since $x_{n_k} \to z_0$ then $Ax_{n_k} \to Az_0$. Consequently, $z_1 = Az_0$. Similarly, $z_2 = Az_1, z_3 = Az_2, z_0 = Az_p$. However, the condition $d(Ax, Ay) < d(x, y)$ implies that there may not be a periodic orbit. Really,

$$d(z_1, z_0) = d(Az_0, Az_1) < d(z_0, z_p) = d(Azp, Az_{p-1}) < \cdots < d(z_2, z_1) = d(Az_1, Az_0) < d(z_1, z_0).$$

The contradiction $d(z_1, z_0) < d(z_1, z_0)$ proves the absence of any periodic orbit. So, $z_0 = z_1 = \ldots = z_p$ is the only fixed point: $z_0 = Az_0$.

17. Prove that a uniformly bounded set of functions in $C[a, b]$, which satisfy the Lipshitz condition with the same common constant, is compact in $C[a, b]$.

$x(t)$ satisfies the Lipshitz condition with constant $L$ if $\forall t, s : |x(t) - x(s)| \leq C|t - s|$. 

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18. Determine whether the following sets in $C[0,1]$ are relatively compact (pre-compact):

(a) $x_n(t) = \sin(nt)$
Not relatively compact: $|\sin(nt)|' = n \cos(nt) = n \to \infty$ at $t = 0$.

(b) $x_n(t) = \sin(t + n)$
This set is pre-compact: $|x_n'(t)| = |\cos(t + n)| \leq 1$. Uniform boundedness of slopes implies equicontinuity.

(c) $x_\alpha(t) = \arctan(\alpha t)$, $\alpha \in \mathbb{R}$
Not pre-compact. Do like in (a)

(d) $x_\alpha(t) = e^{t - \alpha}$, $\alpha \in \mathbb{R}$, $\alpha \geq 0$.
Pre-compact. $|x_\alpha'(t)| = e^{t - \alpha} \leq e$ since $0 \leq t \leq 1$.

**Bonus 2:** Prove that the condition $d(f(x), f(y)) < d(x, y)$, $x \neq y$, is insufficient for the existence of a fixed point of function $f$.

**Solution.** Consider in usual metric $f : [0, \infty) \to [0, \infty)$, $f(x) = x + \frac{1}{x+1}$. The graph $y = f(x)$ is above the bisector $y = x$ and approaches it as $x \to \infty$. Its slope is positive but less than 1 at all points. Hence, $d(f(x), f(y)) < d(x, y)$.


**References**


