Ma 635. Real Analysis I. Hw8, due 11/02.

HW 8 (due 11/02):


Problems:

1. [1] p. 271 # 3
   In the definition of the outer measure let’s consider finite unions of intervals covering the set. Show that if $Q \cap [0, 1]$ is contained in a finite union of open intervals $\bigcup_{i=1}^{n} (a_i, b_i)$ then $\sum_{i=1}^{n} (b_i - a_i) \geq 1$.
   Thus, $Q \cap [0, 1]$ would have measure 1 by this definition.

2. [1] p. 271 # 4
   Given any subset $E$ of $R$ and any $h \in R$, show that $m^*(E + h) = m^*(E)$.

3. [1] p. 271 # 9
   If $E = \bigcup_{n=1}^{\infty} I_n$ is a countable union of pairwise disjoint intervals, prove that $m^*(E) = \sum_{n=1}^{\infty} |I_n|$.

4. [1] p. 271 # 17
   If $m^*(E) = 0$, show that $E^c$ is dense.

5. [1] p. 271 # 18
   If $E$ is a compact set with $m^*(E) = 0$, prove that $\forall \varepsilon > 0 \exists \{I_i\}_{i=1}^{n}$-open intervals, satisfying $\sum_{j=1}^{n} m^*(I_j) < \varepsilon$.

   If $m^*(E) = 0$, prove that $m^*(E^2) = 0$.

7. [1] p. 273 # 22
   Let $E = \bigcup_{n=1}^{\infty} E_n$. Show that $m^*(E) = 0 \iff \forall n m^*(E_n) = 0$.

   Given a bounded open set $G$ and $\varepsilon > 0$, show that there is a compact set $F \subset G$ such that $m^*(F) > m^*(G) - \varepsilon$.

   For each $n$, let $G_n$ be an open subset of $[0, 1]$ containing the rationals in $[0, 1]$ with $m^*(G_n) < 1/n$, and let $H = \bigcap_{n=1}^{\infty} G_n$. Prove that $m^*(H) = 0$ and that $[0, 1] \setminus H$ is a first category set in $[0, 1]$.
   Thus, $[0, 1]$ is the disjoint union of two “small” sets!

10. (bonus) [1] p. 274 # 29

References


