FE610 Stochastic Calculus for Financial Engineers
Lecture 3. Calculus in Deterministic and Stochastic Environments

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Outline

1. Modeling Random Behavior
2. Some Tools of Standard Calculus
3. The Integral
4. Partial Derivatives
5. Total Derivatives
6. Taylor Series Expansion
7. Ordinary Differential Equations
The manner in which information flows in financial markets is more consistent with **stochastic calculus** than with "**standard calculus**".

- For example, the relevant "time interval" may be different on different trading days.
- Numerical methods used in pricing securities are costly in terms of computer time. Hence, the pace of activity may make analysts choose coarser or finer time intervals depending on the level of volatility.

**Some reasons behind developing a new calculus:**

- A complicated random variable can have a very simple structure in continuous time, once the attention is focused on infinitesimal intervals.
- A "binomial" structure may be a good approximation to reality during an infinitesimal interval $dt$, but not necessarily in a large "discrete time" interval denoted by $\Delta$.
- The main tool of stochastic calculus, Ito integral, may be more appropriate to use in financial markets than the Riemann integral used in standard calculus.
Functions:

Suppose $A$ and $B$ are two sets, and let $f$ be a rule which associate to every element $x$ of $A$, exactly one element $y$ in $B$. Such a rule is called a function or a mapping. In mathematical analysis, functions are denoted by

$$f := A \to B$$

(1)

or by

$$y = f(x), \ x \in A.$$  

(2)

If the set $B$ is made of real numbers, then we say that $f$ is a real-valued function and write

$$f := A \to R$$

(3)

If $A$ and $B$ are themselves collections of functions, then $f$ transforms a function into another, and is called an operator.
Random Functions:

\[ y = f(x), \ x \in A, \] (4)

once the value of \( x \) is given, we get the element \( y \). Often \( y \) is assumed to be a real number. Now consider the following significant alternation.

There is a set \( W \), where \( \omega \in W \) denotes a state of the world. The function \( f \) depends on \( x \in R \) and on \( \omega \in W \):

\[ f : R \times W \rightarrow R, \] (5)

or

\[ y = f(x, \omega), \ x \in R, \omega \in W, \] (6)

where the notation \( R \times W \) implies that one has to ”plug in” to \( f(\cdot) \) two variables, one from the set \( W \), and the other from \( R \).
The function $f(x, \omega)$ has the following property: Given a $\omega \in W$, the $f(\cdot, \omega)$ becomes a function of $x$ only. Thus, for different values of $\omega \in W$ we get different functions of $x$. The $f(x, \omega)$ can be called random function or stochastic process.

**Figure : 1** - Randomness of a stochastic process is in terms of the trajectory as a whole.
The Exponential Function

The infinite sum

\[ 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \ldots + \frac{1}{n!} + \ldots \]  

(7)

converges to an irrational number between 2 and 3 as \( n \to \infty \). This number is denoted by the letter \( e \). The exponential function is obtained by raising \( e \) to a power of \( x \):

\[ y = e^x, \, x \in R. \]  

(8)

It has the following properties (discounting asset prices):  

\[ \frac{dy}{dx} = e^{f(x)} \frac{df(x)}{dx}. \]  

(9)

\[ e^x e^y = e^{x+y}. \]  

(10)

Finally, if \( x \) is a random variable, the \( y = e^x \) will be random.
The Logarithmic Function

The logarithmic function is defined as the inverse of the exponential function. Practitioners often work with the logarithm of asset prices (log return). Given

\[ y = e^x, x \in R, \] (11)

the natural logarithm of \( y \) is given by

\[ \ln(y) = x, y > 0. \] (12)

Functions of Bounded Variation

Suppose a time interval is given by [0, \( T \)]. We partition this interval into \( n \) subintervals by selecting the \( t_i, i = 1, \ldots, n \), as

\[ 0 = t_0 \leq t_1 \leq t_2 \leq \ldots \leq t_n = T. \] (13)

The \( [t_i - t_{i-1}] \) represents the length of the \( i \)th subinterval.
Functions of Bounded Variation

Now consider a function of time \( f(t) \), defined on the interval \([0, T]\):

\[
f : [0, T] \rightarrow R. \tag{14}
\]

We form the sum

\[
\sum_{i=1}^{n} |f(t_i) - f(t_{i-1})|. \tag{15}
\]

Given that uncountably many partitions are possible, the sum assumes uncountably many values. If these sums are bounded from above the function \( f(\cdot) \) is said to be of **bounded variation**. It implies functions are not excessively "irregular".

\[
V_0 = \max \sum_{i=1}^{n} |f(t_i) - f(t_{i-1})| < \infty \tag{17}
\]
• **An Example**

Consider function

\[
 f(t) = \begin{cases} 
 tsin\left(\frac{\pi}{t}\right) & \text{when } 0 < t \leq 1 \\
 0 & \text{when } t = 0
\end{cases}
\]  

(18)

It can be shown that \( f(t) \) is not of bounded variation.

**Figure : 2** - Note that as \( t \to 0 \), \( f \) becomes excessively "irregular".
Convergence and Limit

Suppose we are given a sequence

\[ x_0, x_1, x_2, \ldots, x_n, \ldots \]  \hfill (19)

where \( x_n \) represents an object that changes as \( n \) is increased.

In the case where \( x_n \) represents real numbers, we can state this more formally:

**DEFINITION:** We say that a sequence of real numbers \( x_n \) converges to \( x^* < \infty \) if for arbitrary \( \epsilon > 0 \), there exists a \( N < \infty \) such that

\[ |x_n - x^*| < \epsilon \text{ for all } n > N \]  \hfill (20)

We call \( x^* \) the limit of \( x_n \).
The Derivative

The notion of the derivative can be looked at in (at least) two different ways.

- It is a way of defining rates of change of variables under consideration. In particular, if trajectories of asset prices are "too irregular", then their derivative may not exist.
- The derivative is a way of calculating how one variable responds to a change in another variable. For example, given a change in the price of the underlying asset, we may want to know how the market value of an option written on it may move. These types of derivatives are usually taken using the \textit{chain rule}.

**DEFINITION:** Let \( y = f(x) \) be a function of \( x \in \mathbb{R} \). Then the derivative of \( f(x) \) with respect to \( x \), if it exists, is formally denoted by the symbol \( f_x \) and is given by

\[
f_x = \lim_{\Delta \to 0} \frac{f(x + \Delta) - f(x)}{\Delta}
\] (21)
The Derivative (continued)

The variable $x$ can represent any real-life phenomenon. Suppose it represents time. The $\Delta$ would correspond to a finite time interval. The $f(x)$ would be the value of $y$ at time $x$, and the $f(x + \Delta)$ would represent the value of $y$ at time $x + \Delta$.

- The numerator in (21) is the change in $y$ during a time interval $\Delta$. The ratio itself becomes the rate of change in $y$ during the same interval.
- Why is a limit being taken in (21)? It is taken to make the ratio in (21) independent of the size of $\Delta$, the time interval that passes.
- Making the ratio independent of the size of $\Delta$, one pays a price. The derivative is defined for *infinitesimal* intervals. For larger intervals, the derivative becomes an *approximation* that deteriorates as $\Delta$ gets larger and larger.
The Derivative of Exponential Function

Consider the exponential function:

\[ f(x) = Ae^{rx}, \quad x \in R. \]  \hspace{1cm} (22)

\[ f_x = \frac{df(x)}{dx} = r[Ae^{rx}] = rf(x), \quad \text{or} \quad \frac{f_x}{f(x)} = r. \]  \hspace{1cm} (23)

**Figure : 3** - The quantity \( f_x \) is the rate of change of \( f(x) \) at point \( x \). 

\[ \text{Slope} = r(Ae^{rx}) \]
The Derivative as an Approximation

Let $\Delta$ be a finite interval. Then, using the definition in (21) and if $\Delta$ is "small", we can write approximately

$$f(x + \Delta) \approx f(x) + f_x \times \Delta. \quad (24)$$

Figure : 4 - The quality of approximation depends on the size of $\Delta$ and the shape of $f(\cdot)$. 
Figure : 5 - Here, $\Delta$ is large. The approximation $f(x) + f_x \Delta$ is not near $f(x + \Delta)$.

Figure : 6 - When function $f(\cdot)$ is "irregular"/not smooth, the approximation is likely to fail.
Example: High Variation

Consider the case that function $f(x)$ is continuous, but exhibits extreme variations even in small intervals $\Delta$.

$f(x + \Delta) \approx f(x) + f_x \times \Delta$. (25)

Here, not only is the prediction likely to fail, but even a satisfactory definition of $f_x$ may not be obtained.

Figure: When function $f(\cdot)$ is "irregular"/not smooth, the approximation is likely to fail.
The Chain Rule

The second use of the derivative is the chain rule. In the examples discussed earlier, \( f(x) \) was a function of \( x \), and \( x \) was assumed to represent time. The derivative was introduced as the response of a variable to a variation in time.

In pricing derivative securities, we face a somewhat different problem. The price of a derivative asset, e.g., a call option, will depend on the price of the underlying asset, and price of the underlying asset depends on time.

Hence, there is a chain effect. Time passes, new (small) events occur, the price of the underlying asset changes, and this affects the derivative asset’s price. In standard calculus, the tool used to analyze these sorts of chain effects is known as the "chain rule".
The Chain Rule (continued)

Suppose in the example just given $x$ was not itself the time, but a deterministic function of time, denoted by the symbol $t \geq 0$

$$x_t = g(t).$$  \hfill (26)

Then the function $f(\cdot)$ is called a composite function

$$y_t = f(g(t)).$$  \hfill (27)

**DEFINITION:** For $f$ and $g$ defined as above, we have

$$\frac{dy}{dt} = \frac{df(g(t))}{dg(t)} \frac{dg(t)}{dt}. \hfill (28)$$

According to this, the chain rule is the product of two derivatives. 1). The derivative of $f(g(t))$ is taken with respect to $g(t)$. 2). The derivative of $g(t)$ is taken with respect to $t$. 
The Integral

The integral is the mathematical tool used for calculating sums. In contrast to the Σ operator, which is used for sums of a countable number of objects, integrals denote sums of uncountably infinite objects.

The general approach in defining integral is, in a sense, obvious. One would begin with an approximation involving a countable number of objects, and then take some limit and move into uncountable objects. Given that different types of limits, the integral can be defined in various ways.

The Riemann Integral

We are given a deterministic function $f(t)$ of time $t \in [0, T]$. Suppose we are interested in integrating this function over an interval $[0, T]$

$$\int_0^T f(s) ds,$$  

(29)
The Integral (continued)

We partition the interval \([0, T]\) into \(n\) disjoint subintervals \(t_0 = 0 < t_1 < ... < t_n = T\), then consider the approximation

\[
\sum_{i=1}^{n} f\left( \frac{t_i + t_{i-1}}{2} \right)(t_i - t_{i-1}),
\]

(30)

**DEFINITION:** Given that

\[
\max_i |t_i - t_{i-1}| \to 0,
\]

(31)

the Riemann integral will be defined by the limit

\[
\sum_{i=1}^{n} f\left( \frac{t_i + t_{i-1}}{2} \right)(t_i - t_{i-1}) \to \int_0^T f(s)ds,
\]

(32)

Where the limit is taken in a standard fashion.
**The Integral (continued)**

A better approximation can be achieved, if the base of the rectangles is small and the function $f(t)$ is smooth - that is, does not vary heavily in small intervals.

![Diagram showing approximation of integral with rectangles](image)

**Figure**: 8 - When function $f(\cdot)$ is "irregular"/not smooth, the approximation is likely to fail.
The Integral (continued)

In the standard calculus, using different heights for rectangles would not give different integral. But a similar conclusion cannot be reached in stochastic environments.

Figure: 9 - Function $f(\cdot)$ shows step variations, and the approximation is likely to fail.
The Integral (continued)

Suppose $f(W_t)$ is a function of a random variable $W_t$ and that we are interested in calculating

$$\int_{t_0}^{T} f(W_s) dW_s,$$

(33)

The choice of rectangles defined by (where $W_t$ is a martingale)

$$f(W_{t_i})(W_t - W_{t_{i-1}}),$$

(34)

will result in a different expression from the rectangles:

$$f(W_{t_{i-1}})(W_t - W_{t_{i-1}}),$$

(35)

Then the expectation of the term in (36), conditional on information at time $t_{i-1}$, will vanish. This is because the future increments of a martingale will be unrelated to the current information set.
The Integral (continued)

Note that when $f(\cdot)$ depends on a random variable, the resulting integral itself will be a random variable. In this sense, we will be dealing with random integral.

Figure: 9 - In stochastic calculus, different definitions of approximating rectangles may lead to different results.
The Stieltjes Integral

Define differential $df$ as a small variation in the function $f(x)$ due to an infinitesimal variation in $x$

$$df(x) = f(x + dx) - f(x). \quad (36)$$

We have already discussed the equality

$$df(x) = f_x(x) \, dx \quad (37)$$

Now suppose we want to integrate a function $h(x)$ with respect to $x$

$$\int_{x_0}^{x_n} h(x) \, dx \quad (38)$$

where the function $h(x)$ is given by

$$h(x) = g(x) f_x(x). \quad (39)$$
The Stieltjes Integral (continued)

The Stieltjes integral is defined as

$$\int_{x_0}^{x_n} h(x) \, dx,$$

with $df(x) = f_x(x) \, dx$.

This definition is not very different from that of the Riemann integral. If $x$ represents time $t$, the Stieltjes integral over a partitioned interval $[0, T]$ is given by

$$\int_0^T g(s) df(s) \approx \sum_{i=1}^{n} g\left(\frac{t_i + t_{i-1}}{2}\right)(f(t_i) - f(t_{i-1})).$$

Because of these similarities, the limit as $\max_i |t_i - t_{i-1}| \to 0$ of the right-hand side is known as the Riemann-Stieltjes integral.
Example

We let

\[ g(S_t) = aS_t, \quad (44) \]

where \( a \) is a constant. This makes \( g(\cdot) \) a linear function of \( S_t \). What is the value of the integral

\[ \int_0^T aS_t dS(t) = 0.3, \quad (45) \]

If the Riemann-Stieltjes definition is used? Directly taking the integral gives

\[ \int_0^T aS_t dS(t) = a \left[ \frac{1}{2} S_t^2 \right]_0^T \quad (46) \]
The Riemann-Stieltjes Integral - Example

**Example**

Due to the linearity of $g(\cdot)$, the area of the rectangle $S_0ABS_T$

$$a \left[ \frac{S_T + S_0}{2} \right] [S_T - S_0] = a \left[ \frac{1}{2} S_T^2 - \frac{1}{2} S_0^2 \right].$$  \hspace{1cm} (47)

**Figure :** 9 – Due to the linearity of $g(\cdot)$, a single rectangle is sufficient to replicate the area.
Integration by Parts

Consider two differentiable functions $f(t)$ and $h(t)$, where $t \in [0, T]$ represents time. Then it can be shown that

$$
\int_0^T f_t(t)h(t)dt = \left[f(T)h(T) - f(0)h(0)\right] - \int_0^T h_t(t)f(t)dt,
$$

where $h_t(t)$ and $f_t(t)$ are the derivatives of the corresponding functions with respect to time. They are themselves functions of time $t$.

The stochastic version of this transformation is very useful in evaluating Itô integrals. In fact, imagine that $f(\cdot)$ is random while $h(\cdot)$ is (conditionally) a deterministic function of time. Then, we can express stochastic integrals as a function of integrals with respect to a deterministic variable.
Partial Derivatives

Consider a call option, time to expiration affects the price (premium) of the call in two different ways. First, as time passes, the expiration date will approach, and the remaining life of the option gets shorter. This lowers the premium. But at the same time, as time passes, the price of the underlying asset will change. This will also affect the premium. We write

\[ C_t = F(S_t, t) \]  

(50)

where \( C_t \) is the call premium, \( S_t \) is the price of the underlying asset, and \( t \) is time. Now suppose we "fix" the time variable \( t \) and differentiate \( F(S_t, t) \) with respect to \( S_t \). The resulting partial derivative,

\[ \frac{\partial F(S_t, t)}{\partial S_t} = F_S, \]  

(51)
Partial Derivatives (continued)

This effect is an abstraction, because in practice one needs some time to pass before \( S_t \) can change.

The partial derivative with respect to time variable can be defined similarly as

\[
\frac{\partial F(S_t, t)}{\partial t} = F_t,
\]

(52)

Again, this shows the abstract character of the partial derivative. As \( t \) changes, \( S_t \) will change as well. But in taking partial derivatives, we behave as if it is a constant.

Because of this abstract nature of partial derivatives, this type of differentiation cannot be used directly in representing actual changes of asset price in financial markets.

They are useful in taking a total change and then splitting it into components that come from different sources.
Partial Derivatives - Example

Consider a function of two variables:

\[ F(S_t, t) = 0.3S_t + t^2, \]  

(53)

where \( S_t \) is the price of a financial asset and \( t \) is time.

Taking the partial with respect to \( S_t \) involves simply differentiating \( F(\cdot) \) with respect to \( S_t \):

\[ \frac{\partial F(S_t, t)}{\partial S_t} = 0.3, \]  

(54)

Taking the partial with respect to \( t \):

\[ \frac{\partial F(S_t, t)}{\partial t} = 2t. \]  

(55)
Total Derivatives

Let this total change be denoted by the differential $dC_t$. How much of this variation is due to a change in the underlying asset’s prices? How much of the variation is the result of the expiration date getting nearer as time passes? Total differentiation is used to answer such questions.

Let $f(S_t, t)$ be a function of the two variables. The total differential is defined as

$$df = \left[ \frac{\partial F(S_t, t)}{\partial S_t} \right] dS_t + \left[ \frac{\partial F(S_t, t)}{\partial t} \right] dt.$$  \hfill (56)

As $t$ changes, $S_t$ will change as well. But in taking partial derivatives, we behave as if it is a constant.

Because of this abstract nature of partial derivatives, this type of differentiation cannot be used directly in representing actual changes of asset price in financial markets.
Taylor Series Expansion

Let $f(x)$ be a infinitely differentiable function of $x \in R$. And pick an arbitrary value of $x$; call this $x_0$.

**DEFINITIONS:** The Taylor series expansion of $f(x)$ around $x_0 \in R$ is defined as

$$f(x) = f(x_0) + f_x(x_0)(x - x_0) + \frac{1}{2}f_{xx}(x_0)(x - x_0)^2$$

$$+ \frac{1}{3!}f_{xxx}(x_0)(x - x_0)^3 + ... = \sum_{i=0}^{\infty} \frac{1}{i!}f^i(x_0)(x - x_0)^i, \quad (57)$$

where $f^i(x_0)$ is the $i$th order derivative of $f(x)$ with respect to $x$ evaluated at the point $x_0$.

We are not going to elaborate on why the expansion is valid, if $f(x)$ is continuous and smooth enough. Taylor series expansion is taken for granted.
Consider the exponential function where $t$ denotes time, $T$ is fixed, $r > 0$ and $t \in [0, T]$:

$$B_t = 100e^{-r(T-t)},$$  \hspace{1cm} (58)

This function begins at $t = 0$ with a value of $B_0 = 100e^{-rT}$. Then it increases at a constant percentage rate $r$. As $t \rightarrow T$, the value of $B_t$ approaches 100.

A first-order Taylor series expansion around $t = t_0$ will be

$$B_t \approx 100e^{-r(T-t_0)} + (r)100e^{-r(T-t_0)}(t - t_0), \quad t \in [0, T],$$  \hspace{1cm} (59)

*Taylor series expansion of $B_t$ shows that, as interest rates increase (decreases), the value of the bound decreases (increases).*
**Ordinary Differential Equations**

The third major notion from standard calculus that we would like to review is the concept of an ordinary differential equation (ODE).

For example, consider the expression

\[ dB_t = -r_t B_t dt \]  
with known \( B_0, r_t > 0 \).

This expression states that \( B_t \) is a quantity that varies with \( t \) - i.e., changes in \( B_t \) are a function of \( t \) and of \( B_t \). The equation is called an *ordinary differential equation*. Here, the percentage variation in \( B_t \) is proportional to some factor \( r_t \) times \( dt \):

\[ \frac{dB_t}{B_t} = -r_t dt. \]

Now, we say that the function \( B_t \) defined by
Ordinary Differential Equations (continued)

\[ B_t = e^{\int_0^t r_u \, du}, \quad (62) \]

solves the ODE in (52) in that plugging it into (54) satisfies the equality (52). Thus, an ordinary differential equation is first of all an equation where there exist one or more unknowns that need to be determined.

\[ dB_t = -r_t B_t \, dt. \quad (63) \]

the solution, with the condition \( B_T = 1 \) was

\[ B_t = e^{\int_0^t r_u \, du}, \quad (64) \]

This example shows that the pricing functions for fixed income securities can be characterized as solutions of some appropriate differential equations.
## ODE - Example

### Example

In a simple equation $3x + 1 = x$, the unknown is $x$, a number to be determined. Here the solution is $x = -1/2$.

In a matrix equation $Ax - b = 0$, the unknown element is a vector. Under appropriate conditions, the solution would be $x = A^{-1}b$.

In an ordinary differential equation,

$$\frac{dx_t}{dt} = -ax_t + b,$$

where the unknown is $x_t$, a function. More precisely, it is a function of $t$ : $x_t = f(t)$. 