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Continuous Time Martingales

Using different information sets, one can conceivably generate different "forecasts" of a process \( \{ S_t \} \). These forecasts are expressed using conditional expectations. In particular,

\[
E_t[S_T] = E[S_T|I_t], \quad t < T, \tag{1}
\]

is the formal way of denoting the forecast of a future value, \( S_T \) of \( S_t \), using the information available as of time \( t \). \( E_u[S_T], u < t \), would denote the forecast of the same variable using a smaller information set as of or earlier than time \( u \).

**DEFINITION:** We say that a process \( \{ S_t, t \in [0, \infty] \} \) is a martingale with respect to the family of information sets \( I_t \) and with respect to the probability \( P \), if, for all \( t > 0 \),
Continuous Time Martingales (continued)

1. $S_t$ is known, give $I_t$. ($S_t$ is $I_t$-adapted.)

2. Unconditional “forecasts” are finite:

$$E|S_t| < \infty.$$  \hspace{1cm} (2)

3. And if

$$E_t[S_T] = S_t, \text{ for all } t < T,$$  \hspace{1cm} (3)

with probability 1. That is, the best forecast of unobserved future values is the last observation on $S_t$.

Here, all expectations $E[\cdot], E_t[\cdot]$ are assumed to be taken with respect to the probability $P$.

[According to this definition, martingales are random variables whose future variations are completely unpredictable given the current information set.]
The Use of Martingales in Asset Pricing

Now, we know that stock prices or bond prices are not completely unpredictable. The price of a discount bond is expected to *increase* over time. In general, the same is true for stock prices. They are expected to increase on average. Hence, if $B_t$ represents the price of a discount bond maturing at time $T$, $t < T$,

$$B_t < E_t[B_u], t < u < T.$$  \hspace{1cm} (4)

Clearly, the price of a discount bond is not a martingale.

- Similarly, in general, a risky stock $S_t$ will have a positive expected return will not be a martingale. For a small interval $\Delta$, we can write

$$E_t[S_{t+\Delta} - S_t] \approx \mu\Delta,$$  \hspace{1cm} (5)

where $\mu$ is a positive rate of expected return.
- It turns out that although most financial assets are not martingales, one can convert them into martingales. For example, one can find a probability distribution $\tilde{\mathbb{P}}$ such that bond or stock prices discounted by the risk-free rate become martingales. If this is done, equalities such as

$$E_{t}^{\tilde{\mathbb{P}}}[e^{-ru}B_{t+u}] = B_{t}, \quad t < u < T - t.$$  

for bonds, or

$$E_{t}^{\tilde{\mathbb{P}}}[e^{-ru}S_{t+u}] = S_{t}, \quad 0 < u.$$  

for stock prices, can be very useful in pricing derivative securities.

- Two ways of converting submartingales into martingales:

  * The first method should be obvious. We can subtract an expected trend from $e^{-rt}S_{t}$ or $e^{-rt}B_{t}$. This would make the deviations around the trend completely unpredictable.
** The second method is to transform its probability distribution. That is, if one had

$$E_t^P[e^{-ru}S_{t+u}] > S_t, \ 0 < u.$$  \hfill (8)

where $E_t^P[\cdot]$ is the conditional expectation calculated using a probability distribution $P$, we may try to find an “equivalent” probability $\tilde{P}$, such that the new expectations satisfy

$$E_t^{\tilde{P}}[e^{-ru}S_{t+u}] = S_t, \ 0 < u.$$  \hfill (9)

and the $e^{-rt}S_t$ becomes a martingale. The probability distributions that convert equation above into equality are called equivalent martingale measures - A transformation based on Girsanov Theorem is more promising than the Doob-Meyer decompositions.
Relevance of Martingales in Stochastic Modeling

Let $X_t$ represent an asset price that has the martingale property with respect to the filtration $\{I_t\}$ and with respect to the probability $\tilde{P}$,

$$E^{\tilde{P}}[X_{t+\Delta}|I_t] = X_t,$$  \hspace{1cm} (10)

where $\Delta > 0$ represents a small time interval. What type of trajectories would such an $X_t$ have in continuous time?

To answer this question, first define the martingale difference $\Delta X_t$,

$$\Delta X_t = X_{t+\Delta} - X_t,$$  \hspace{1cm} (11)

and then note that since $X_t$ is a martingale,

$$E^{\tilde{P}}[\Delta X_t|I_t] = 0.$$  \hspace{1cm} (12)
Relevance of Martingales in Stochastic Modeling (continued)

- As mentioned earlier, this equality implies that increments of a martingale should be totally unpredictable, no matter how small the time interval $\Delta$ is. But, since we are working with continuous time, we can indeed consider very small $\Delta$’s. Martingales should then display very irregular trajectories. In fact, $X_t$ should not display any trends discernible by inspection, even during infinitesimally small time intervals.

- Such irregular trajectories can occur in two different ways. They can be continuous, or they can display jumps. The former leads to continuous martingales, whereas the latter are called right continuous martingales.

- Suppose one is dealing with a continuous martingale $X_t$ that also has finite second moment $E[X_t^2] < \infty$ for all $t > 0$. Such a process is called a continuous square integrable martingale. It is very close to Brownian motion.
Example

Here, we will construct a martingale using two independent Poisson processes observed during “small intervals” $\Delta$.

Suppose financial markets are influenced by “good” and “bad” news. We ignore the content of the news, but retain the information on whether it is “good” or “bad”.

The $N_t^G$ and $N_t^B$ denote the total number of instances of “good” and “bad” news, respectively, until time $t$. We assume further that the way news arrives in financial markets is totally unrelated to past data, and that the “good” and “bad” news are independent.

Finally, during a small interval $\Delta$, at most one instance of good news or bad news can occur, and the probability of this occurrence is the same for both types of news. Thus, the probabilities of incremental changes $\Delta N_t^G, \Delta N_t^B$ during $\Delta$ is assumed to be given approximately by
Example

\[ P(\Delta N_t^G = 1) = P(\Delta N_t^B = 1) \cong \lambda \Delta. \quad (13) \]

Then the variable \( M_t \), defined by

\[ M_t = N_t^G - N_t^B, \quad (14) \]

will be a martingale. Note that the increments of \( M_t \) over small intervals \( \Delta \) will be given by

\[ \Delta M_t = \Delta N_t^G - \Delta N_t^B. \quad (15) \]

Apply the conditional expectation operator:

\[ E_t[\Delta M_t] = E_t[\Delta N_t^G] - E_t[\Delta N_t^B]. \quad (16) \]

But, approximately,

\[ E_t[\Delta N_t^G] \cong 0 \cdot (1 - \lambda \Delta) + 1 \cdot \lambda \Delta \cong \lambda \Delta, \quad (17) \]
Similarly, we get

\[ E_t[\Delta N_t^B] \cong 0 \cdot (1 - \lambda \Delta) + 1 \cdot \lambda \Delta \cong \lambda \Delta, \quad (18) \]

This means that

\[ E_t[\Delta M_t] \cong \lambda \Delta - \lambda \Delta = 0. \quad (19) \]

Hence, increments in \( M_t \) are unpredictable given the family \( I_t \). However, if we assume that “good” news can occur with a slightly greater probability than “bad” news,

\[ P(\Delta N_t^G = 1) = \lambda^G \Delta > P(\Delta N_t^B = 1) \cong \lambda^B \Delta. \quad (20) \]

Then \( M_t \) will cease to be martingale (but a submartingale), since

\[ E_t[\Delta M_t] \cong \lambda^G \Delta - \lambda^B \Delta > 0. \quad (21) \]
Assume that \( \{X_t\} \) represents a trajectory of a continuous square integrable martingale. Pick a time interval \([0, T]\) and consider the times \( \{t_i\} \):

\[
 t_0 = 0 < t_1 < t_2 < \ldots < t_{n-1} < t_n = T.
\]

We define the variation of the trajectory as

\[
 V_1 = \sum_{i=1}^{n} |X_{t_i} - X_{t_{i-1}}| \tag{22}
\]

Heuristically, \( V_1 \) can be interpreted as the length of the trajectory followed by \( X_t \) during the interval \([0, T]\).

The \textit{quadratic} variation is given by

\[
 V_2 = \sum_{i=1}^{n} |X_{t_i} - X_{t_{i-1}}|^2 \tag{23}
\]
Properties of Martingale Trajectories (continued)

One can similarly define higher-order variations. For example, the fourth-order variation is defined as:

\[ V^4 = \sum_{i=1}^{n} |X_{t_i} - X_{t_{i-1}}|^4 \]  \hspace{1cm} (24)

- Remember that we want \( X_t \) to be continuous and to have a nonzero variance. This means two things:
  
  First, as the partitioning of the interval \([0, T]\) gets finer and finer, “consecutive” \( X_t \)'s get nearer and nearer, for any \( \epsilon > 0 \)

  \[ P(|X_{t_i} - X_{t_{i-1}}| > \epsilon) \rightarrow 0, \]  \hspace{1cm} (25)

  Second, as the partitions get finer and finer, we still want

  \[ P(\sum_{i=1}^{n} |X_{t_i} - X_{t_{i-1}}|^2 > 0) = 1. \]  \hspace{1cm} (26)
Properties of Martingale Trajectories (continued)

Now, consider some properties of $V^1$ and $V^2$.

- First, note that even though $X_t$ is a continuous martingale, and $X_{t_i}$ approaches $X_{t_{i-1}}$ as the subinterval $[t_i - t_{i-1}]$ becomes smaller and smaller, this does not mean that $V^1$ also approaches zero. The reader may find this surprising. After all, $V^1$ is made of the sum of such incremental changes:

$$V^1 = \sum_{i=1}^{n} |X_{t_i} - X_{t_{i-1}}|$$

(27)

Surprisingly, the opposite is true. As $[0, T]$ is partitioned into finer and finer subintervals, changes in $X_t$ get smaller. But, at the same time, the number of terms in the sum defining $V^1$ increases. It turns out that in the case of a continuous-time martingale, the second effect dominates and the $V^1$ goes toward infinity.
This can be shown heuristically as follows. We have

\[ \sum_{i=1}^{n} |X_{t_i} - X_{t_i-1}|^2 < \left[ \max_{i} |X_{t_i} - X_{t_i-1}| \right] \sum_{i=1}^{n} |X_{t_i} - X_{t_i-1}| \quad (28) \]

because the right-hand side is obtained by factoring out the “largest” \( |X_{t_i} - X_{t_i-1}| \). This means that

\[ V^2 < \left[ \max_{i} |X_{t_i} - X_{t_i-1}| \right] V^1 \quad (29) \]

As \( t_i \to t_{i-1} \) for all \( i \) the continuity of the martingale implies that “consecutive” \( X_{t_i} \)’s will get very near each other. At the limit,

\[ \max_{i} |X_{t_i} - X_{t_{i-1}}| \to 0. \quad (30) \]

It implies that we must have \( V^1 \to \infty \).
Now consider the same property for higher-order variations. Consider $V^4$ and apply the same “trick” as before:

$$V^4 < \left[ \max_i |X_{t_i} - X_{t_{i-1}}|^2 \right] V^2 \tag{31}$$

As long as $V^2$ converges to a well-defined random variable, the right-hand side of (31) will go to zero. At the limit,

$$\max_i |X_{t_i} - X_{t_{i-1}}|^2 \to 0. \tag{32}$$

It implies that we must have $V^4 \to 0$. Here we summarize the three properties of the trajectories:

1. The variation $V^1$ will converge to infinity in some probabilistic sense and the continuous martingale will behave very irregularly.
Properties of Martingale Trajectories (continued)

2. The quadratic variation $V^2$ will converge to some well-defined random variable.

3. All higher-order variations will vanish in some probabilistic sense.

These properties have important implications. First, we see that $V^1$ is not a very useful quantity to use in the calculus of continuous square integrable martingale, whereas the $V^2$ can be used in a meaningful way. Second, higher-order variations can be ignored if one is certain that the underlying process is a continuous martingale.

Furthermore, since the Riemann-Stieltjes integral uses the equivalent of $V^1$ in deterministic calculus and considers finer and finer partitions of interval under consideration. In stochastic environments such limits do not converge. Instead, stochastic calculus is forced to use $V^2$. 
Example: Brownian Motion

- Suppose $X_t$ represents a continuous process whose increments are normally distributed. Such a process is called a (generalized) Brownian motion. We observe a value of $X_t$ for each $t$. At every instant, the infinitesimal changes in $X_t$ is denoted by $dX_t$. Incremental changes in $X_t$ are assumed to be independent across time.

Under these conditions, if $\Delta$ is small interval, the increments $\Delta X_t$ during $\Delta$ will have a normal distribution with mean $\mu \Delta$ and variance $\sigma^2 \Delta$. This means

$$\Delta X_t \sim N(\mu \Delta, \sigma^2 \Delta). \quad (33)$$

The fact that increments are uncorrelated can be expressed as

$$E[(\Delta X_u - \mu \Delta)(\Delta X_t - \mu \Delta)] = 0, \ u \neq t. \quad (34)$$
Example

Example: Brownian Motion

- Leaving aside formal aspects of defining such a process $X_t$ here we ask a simple question: is $X_t$ a martingale?

The process $X_t$ is the “accumulation” of infinitesimal increments over time, that is,

$$X_{t+T} = X_0 + \int_0^{t+T} dX_u.$$  \hspace{1cm} (35)

Assuming that the integral is well defined, we can calculate the relevant expectations. Consider the expectation take with respect to the probability distribution given in (33), and given the information on $X_t$ observed up to time $t$:

$$E[X_{t+T}] = E_t[X_t + \int_t^{t+T} dX_u].$$  \hspace{1cm} (36)
Example

Examples of Martingales (continued)

But at time $t$, future values of $\Delta X_{t+T}$ are predictable because all changes during small intervals $\Delta$ have expectation equal to $\mu \Delta$. This means

$$E_t\left[\int_t^{t+T} dX_u\right] = \mu T.$$ \hspace{1cm} (37)

So, $E[X_{t+T}] = X_t + \mu T$. \hspace{1cm} (38)

Clearly, $\{X_t\}$ is not a martingale with respect to the distribution in Eq. (33) and with respect to the information on current and past $X_t$. But, this last result gives a clue to how to generate a martingale with $\{X_t\}$. Consider the process:

$$Z_t = X_t - \mu t.$$ \hspace{1cm} (39)
It is easy to show that $Z_t$ is a martingale:

$$E[Z_{t+T}] = E[X_{t+T} - \mu(t + T)]$$

$$= E[X_t + (X_{t+T} - X_t)] - \mu(t + T)].$$

which means

$$E[Z_{t+T}] = X_t + E[(X_{t+T} - X_t)] - \mu(t + T).$$

But the expectation on the right-hand side is equal to $\mu T$, as shown in Eq. (38). This means

$$E[Z_{t+T}] = X_t - \mu t = Z_t$$

That is, $Z_t$ is a martingale. Hence, we were able to transform $X_t$ into a martingale by subtracting a deterministic function.
In this section, we formalize these special cases and discuss the so-called Doob-Meyer decomposition.

Suppose a trader observes at times $t_i$, $t_0 < t_1 < t_2 < \ldots < t_{n-1} < t_k = T$, the price of a financial asset $S_t$.

If the intervals between the times $t_{i-1}$ and $t_i$ are very small, and if the market is "liquid", the price of the asset is likely to exhibit at most one uptick or one downtick during a typical $t_i - t_{i-1}$. We formalize this by saying that at each instant $t_i$, there are only two possibilities for $S_{t_i}$ to change:

$$\Delta S_{t_i} = \begin{cases} 1 & \text{with probability } p \\ -1 & \text{with probability } (1-p) \end{cases}$$

(44)

It is assumed that these changes are independent of each other. Further, if $p = 1/2$, then the expected value of $\Delta S_{t_i}$ will equal to zero, otherwise nonzero.
Given that a typical object of interest is a sample path, or trajectory, of price changes, we first need to construct a set made of all possible paths. This space is called a sample space. Its elements are made of sequences of +1’s and −1’s. For example, a typical sample path can be

\[ \{ \Delta S_{t_i} = -1, \ldots, \Delta S_{t_k} = +1 \}, \quad (45) \]

Since \( k \) is finite, given an initial point \( S_{t_0} \), we can easily determine the trajectory followed by the asset price by adding incremental changes. This way we can construct the set of all possible trajectories, i.e., the sample space.

For example, the particular sequence of \( \Delta S^* \) that begins with +1 at time \( t_0 \) and alternates until time \( t_k \),

\[ \Delta S^* = \{ \Delta S_{t_1} = +1, \Delta S_{t_2} = -1, \ldots, \Delta S_{t_k} = -1 \}, \quad (46) \]
- We will have the probability (assuming $k$ is even)

$$P(S^*) = p^{k/2}(1 - p)^{k/2}. \quad (47)$$

Since $k$ is finite, there are a finite number of possible trajectories in the sample space, and we can assign a probability to every one of these trajectories.

- Another assumption that simplifies this task is the independence of successive price changes. This way, the probability of the whole trajectory can be obtained by simply multiplying the probabilities associated with each incremental change. One can easily obtain the level of the asset price from subsequent changes, given the opening price $S_{t_0}$:

$$S_{t_t} = S_{t_0} + \sum_{i=1}^{k} (S_{t_i} - S_{t_{i-1}}). \quad (48)$$
Martingale Representations (continued)

- The highest possible value for $S_{tk}$ is $S_{t0} + k$. This value will result if all incremental changes $\Delta S_{tk}, i = 1, ..., k$ are made of $+1$'s. The probability of this outcome is

$$P(S_{tk} = S_{t0} + k) = p^k. \quad (49)$$

Similarly, the lowest possible value of $S_{tk}$ is $S_{t0} - k$. The probability of this is given by

$$P(S_{tk} = S_{t0} - k) = (1 - p)^k. \quad (50)$$

- In general, the price would be somewhere between these two extremes. Of the $k$ incremental changes observed, $m$ would be made of $+1$'s and $k - m$ made of $-1$'s, with $m \leq k$. The $S_{tk}$ will assume the value

$$S_{tk} = S_{t0} + m - (k - m). \quad (51)$$

Note: several trajectories may result in the same value.
Martingale Representations (continued)

- Adding the probabilities associated with all these combinations, we obtain

\[ P(S_{t_k} = S_{t_0} + 2m - k) = C_k^{k-m} p^m (1 - p)^{k-m}, \quad (52) \]

\[ C_k^{k-m} = \frac{k!}{m!(k-m)!}. \quad (53) \]

- This probability is given by the binomial distribution. As \( k \to \infty \), this distribution converges to normal distribution. Consider the expectation under the probabilities in (53).

\[ E^p[(S_{t_k} | S_{t_0}, \Delta S_{t_1}, ..., \Delta S_{t_{k-1}}] = S_{t_{k-1}} + [(+1)p + (-1)(1 - p)], \quad (54) \]

where the second term on the right-hand side is the expectation of \( \Delta S_{t_k} \), the unknown increments given the information at time \( I_{t_{k-1}} \). Clearly, if \( p = 1/2 \), this term is zero, and we have a martingale.
Martingale Representations (continued)

- The \( \{S_{t_k}\} \) will be martingale with respect to the information set generated by past price changes and with respect to this particular probability distribution

\[
E^p[(S_{t_k} | S_{t_0}, \Delta S_{t_1}, ..., \Delta S_{t_{k-1}}) = S_{t_{k-1}} \quad (55)
\]

- If \( p \neq 1/2 \), the \( \{S_{t_k}\} \) will cease to be a martingale with respect to \( \{I_{t_k}\} \). However, the centered process \( \{Z_{t_k}\} \), defined by

\[
Z_{t_k} = [S_{t_0} + (1 - 2p)] + \sum_{i=1}^{k} [\Delta S_{t_i} + (1 - 2p)] \quad (56)
\]

or

\[
Z_{t_k} = S_{t_k} + (1 - 2p)(k + 1) \quad (57)
\]

will again be a martingale with respect to \( I_{t_k} \)
Consider the case where the probability of an uptick at any time $t_i$ is somewhat greater than the probability of a downtick for a particular asset, so that we expect a general upward trend in observed trajectories: $1 > p > 1/2$.

Then, as shown earlier,

$$E^P[S_{t_k} | S_{t_0}, S_{t_1}, ..., S_{t_{k-1}}] = S_{t_{k-1}} - (1 - 2p),$$  \hspace{1cm} (58)

which means,

$$E^P[S_{t_k} | S_{t_0}, S_{t_1}, ..., S_{t_{k-1}}] > S_{t_{k-1}},$$  \hspace{1cm} (59)

since $2p > 1$. This implies that $\{S_{t_k}\}$ is a submartingale.

Now, as shown earlier, we can write

$$S_{t_{k-1}} = -(1 - 2p)(k + 1) + Z_{t_k},$$  \hspace{1cm} (60)

where $Z_{t_k}$ is a martingale.
### The General Case

- The decomposition of an upward-trending submartingale into a deterministic trend and a martingale component was done for a process observed at a finite number of points during a continuous interval.

- The Doob-Meyer theorem provides the answer to this question. We state the theorem without proof.

Let \{I_t\} be the family of information sets discussed above.

**THEOREM:** If \(X_t, 0 \leq t \leq \infty\) is a right-continuous submartingale with respect to the family \{I_t\} and if \(E[X_t] < \infty\) for all \(t\), then \(X_t\) admits

\[
X_t = M_t + A_t,
\]

(61)

where \(M_t\) is a right-continuous martingale with respect to probability \(P\), and \(A_t\) is an increasing process measurable with respect to \(I_t\).
The Use of Doob-Meyer Decomposition

- The Doob-Meyer decomposition shows that even if continuously observed asset prices contain occasional jumps and trend upwards at the same time, then we can convert them into martingales by subtracting a process observed at time $t$.

- We assume again that time $t \in [0, T]$ is continuous. The value of a call option $C_t$ written on the underlying asset $S_t$ will be given by the function

$$C_T = \max[S_T - K, 0]$$  \hspace{1cm} (62)

at expiration date $T$. At an earlier time $t, t < T$, the exact value of $C_T$ is unknown. But we can calculate a forecast of it using the information $I_t$ available at time $t$,

$$E^P[C_T | I_t] = E^P[\max[S_T - K, 0] | I_t]$$  \hspace{1cm} (63)
The Use of Doob-Meyer Decomposition (continued)

- Given this forecast, one may be tempted to ask if the fair market value $C_t$ will equal a properly discounted value of $E^P[max[S_T - K, 0]|I_t]$.

- For example, suppose we use the (constant) risk-free interest rate $r$ to discount $E^P[max[S_T - K, 0]|I_t]$, to write

$$C_T = e^{-r(T-t)}E^P[max[S_T - K, 0]|I_t].$$

(64)

Would this equation give the fair value $C_t$ of the call option? The answer depends on whether or not $e^{-rt}C_t$ is a martingale with respect to the pair $I_t, P$. If it is, we have

$$E^P[e^{-rt}C_T|I_t] = e^{-rt}C_t, t < T,$$

(65)

Then $e^{-rt}C_t$ will be a martingale.

But can we expect $e^{-rt}S_t$ to be martingale under the true probability $P$?
The Use of Doob-Meyer Decomposition (continued)

- As discussed earlier, under the assumption that investors are risk-averse, for a typical risky security we have

\[ C_T = E^P [e^{-r(T-t)} S_T | S_t] > S_t. \]  \hspace{1cm} (66)

That is \( e^{-rt} S_t \) will be a submartingale.

- But, according to Doob-Meyer decomposition, we can decompose the \( e^{-rt} S_t \) to obtain

\[ e^{-rt} S_t = A_t + Z_t, \quad t < T, \]  \hspace{1cm} (67)

where \( A_t \) is an increasing \( I_t \) measurable random variable, and \( Z_t \) is a martingale with respect to the information \( I_t \).

- However, in practice, it is more convenient and significantly easier to convert asset prices into martingales, not by subtracting their drift, but instead by changing the underlying probability distribution \( P \).
The First Stochastic Integral

- Let $H_{t_{i-1}}$ be any random variable adapted to $I_{t_{i-1}}$. Let $Z_t$ be any martingale with respect to $I_t$ and to some probability measure $P$. Then the process defined by

$$M_{tk} = M_{t0} + \sum_{i=1}^{k} H_{t_{i-1}} [Z_{ti} - Z_{t_{i-1}}]$$

(68)

will also be a martingale with respect to $I_t$.

- $Z_t$ is a martingale and has unpredictable increments. The fact that $H_{t_{i-1}}$ is $I_{t_{i-1}}$-adapted means $H_{t_{i-1}}$ are “constants” given $I_{t_{i-1}}$. Then, increments in $Z_{ti}$ will be uncorrelated with $H_{t_{i-1}}$ as well. Using these observations, we can calculate

$$E_{t_0}[M_{tk}] = M_{t0} + E_{t_0}\left[\sum_{i=1}^{k} E_{t_{i-1}}[H_{t_{i-1}}(Z_{ti} - Z_{t_{i-1}})]\right].$$

(69)
The First Stochastic Integral (continued)

But increments in $Z_{t_i}$ are unpredictable as of time $t_{i-1}$. Also, $H_{t_{i-1}}$ is $I_t$-adapted. This means we can move the $E_{t_{i-1}}[\cdot]$ operator “inside” to get

$$H_{t_{i-1}} E_{t_{i-1}}[Z_{t_i} - Z_{t_{i-1}}] = 0. \quad (70)$$

This implies

$$E_{t_0}[M_{t_k}] = M_{t_0}. \quad (71)$$

$M_t$ thus has the martingale property.

It turns out that $M_t$ defined this way is the first example of a **stochastic integral**. The questions is whether we can obtain a similar results when $\sup_i [t_i - t_{i-1}]$ goes to zero. Using some analogy, we can obtain an expression

$$M_t = M_0 + \int_0^t H_u dZ_u. \quad (72)$$
Application to Finance: Trading Gains

- We consider a decision maker who invests in both a riskless and a risky security at trading time \( t_i \):

\[
0 = t_0 < \ldots < t_i < \ldots < t_n = T. \tag{73}
\]

Let \( \alpha_{t_{i-1}} \) and \( \beta_{t_{i-1}} \) be the number of shares of riskless and risky securities held by the investor right before time \( t_i \) trading begins. Clearly, these random variables will be \( I_{t_i} \)-adapted. \( \alpha_{t_i} \) and \( \beta_{t_i} \) are the nonrandom initial holdings.

- Suppose we now consider trading strategies that are \textit{self-financing}. These are strategies where time \( t_i \) investments are financed solely from the proceeds of time \( t_{i-1} \) holdings. That is, they satisfy

\[
\alpha_{t_{i-1}} B_{t_i} + \beta_{t_{i-1}} S_{t_i} = \alpha_{t_i} B_{t_i} + \beta_{t_i} S_{t_i}, \tag{74}
\]

where \( i = 1, 2, \ldots, n \).
Application to Finance: Trading Gains (continued)

- According to this strategy, the investor can sell his holdings at time $t_i$ for an amount equal to the left-hand side of the equation, and with all of these proceeds purchase $\alpha_{t_i}$ and $\beta_{t_i}$ units of riskless and risky securities. We can now substitute recursively for the left-hand side using Eq (74) for $t_{i-1}, t_{i-2}, \ldots$, and using the definitions

$$B_{t_i} = B_{t_i-1} + [B_{t_i} - B_{t_i-1}] \quad (75)$$
$$S_{t_i} = S_{t_i-1} + [S_{t_i} - S_{t_i-1}] \quad (76)$$

We obtain

$$\alpha_{t_0} B_{t_0} + \beta_{t_0} S_{t_0} + \sum_{j=1}^{i-1} [\alpha_{t_j} [B_{t_i} - B_{t_i-1}] + \beta_{t_j} [S_{t_i} - S_{t_i-1}]]$$

$$= \alpha_{t_i} B_{t_i} + \beta_{t_i} S_{t_i} \quad (77)$$

where the RHS is the wealth of the decision maker after time $t_i$ trading.