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Translation of Probabilities

Recent methods of derivative asset pricing do not necessarily exploit PDEs implied by arbitrage-free portfolios. They rest on converting prices of such assets into martingales.

This is done through transforming the underlying probability distributions using the tools provided by the Girsanov theorem. The Girsanov theorem provides the general framework for transforming one probability measure into another “equivalent” measure in more complicated sense. The theorem covers the case of Brownian motion. Hence, the state space is continuous, and the transformations are extended to continuous-time stochastic processes.
The Girsanov Theorem

- The general method can be summarized as follows:
  1. We have an expectation to calculate.
  2. We transform the original probability measure so that expectation becomes easier to calculate.
  3. We calculate the expectation under the new probability.
  4. Once the result is calculated and if desired, we transform this probability back to the original distribution.

- We are given a family of information sets \( \{I_t\} \) over a period of \([0, T]\). \( T \) is finite. We define a random process \( \xi_t \):

\[
\xi_t = e^{(\int_0^t X_u dW_u - \frac{1}{2} \int_0^t X_u^2 du)}, \quad t \in [0, T],
\]

(1)

where \( X_t \) (a Wiener process with probability \( P \)) is an \( I_t \)-measurable process. We impose an additional condition on \( X_t \) (Novikov condition):

\[
E[\int_0^t X_u^2 du] < \infty, \quad t \in [0, T].
\]

(2)
The Girsanov Theorem (continued)

- It turns out that if the Novikov condition is satisfied, the $\xi_t$ will be a square integrable martingale. Using Ito’s Lemma, calculate the differential

$$d \xi_t = [e^{\left(\int_0^t X_u dW_u - \frac{1}{2} \int_0^t X_u^2 du\right)}][X_t dW_t],$$

which reduces to

$$d \xi_t = \xi_t[X_t dW_t],$$

(4)

Also, we see by simple substitution of $t = 0$ in the random process $\xi_t$

$$\xi_0 = 1.$$  

(5)

- Thus, by taking the stochastic integral, we obtain

$$\xi_t = 1 + \int_0^t \xi_s X_s dW_s.$$  

(6)
But the term $\int_0^t \xi_s X_s dW_s$ is a stochastic integral with respect to a Wiener process. Also, the term $\xi_s X_s$ is $I_t$-adapted and does not move rapidly. All these imply, as shown before, that the integral is a (square integrable) martingale,

$$E \left[ \int_0^t \xi_s X_s dW_s \right] = \int_0^u \xi_s X_s dW_s, \; u < t \tag{7}$$

- **THEOREM:** If the process $\xi_t$ defined in (1) is a martingale with respect to information sets $I_t$, and the probability $P$, then $\tilde{W}_t$ defined by

$$\tilde{W}_t = W_t - \int_0^t X_u du, \; t \in [0, T], \tag{8}$$

is a Wiener process with respect to $I_t$ and with respect to the probability measure $\tilde{P}_T$, given by
The Girsanov Theorem (continued)

\[ \tilde{P}_T = E^P[1_A\xi_T], \quad (9) \]

with \( A \) being an event determined by \( I_T \) and \( 1_A \) being the indicator function of the event.

- In heuristic terms, this theorem states that if we are given a Wiener process \( W_t \), then, multiplying the probability distribution of this process by \( \xi_t \), we can obtain a new Wiener process \( \tilde{W}_t \) with probability distribution \( \tilde{P} \). The two processes are related to each other through

\[ d\tilde{W}_t = dW_t - X_t dt. \quad (10) \]

That is, \( \tilde{W}_t \) is obtained by subtracting an \( I_t \)-adapted drift from \( W_t \). The main condition for performing such transformations is that \( \xi_t \) is a martingale with \( E[\xi_T] = 1 \).
A Discussion of the Girsanov Theorem

- Suppose the $X_u$ was constant and equaled $\mu$:

$$X_u = \mu. \quad (11)$$

Then, taking the integrals in the exponent in a straightforward fashion, and remembering that $W_0 = 0$,

$$\xi_t = e^{\frac{1}{2} \sigma^2 [\mu W_t - \frac{1}{2} \mu^2 t]}, \quad (12)$$

which is similar to the $\xi(z_t)$ discussed earlier. This shows the following:

1. The symbol $X_t$ used in the Girsanov theorem plays the same role $\mu$ played in simpler settings. It measures how much the original “mean” will be changed.

2. In earlier examples, $\mu$ was time independent. Here $X_t$ may depend on any random quantity, as long as this random quantity is known by time $t$ ($I_t$-adapted).

3. The $\xi_t$ is a martingale with $E[\xi_t] = 1$. 
A Discussion of the Girsanov Theorem (continued)

- Consider the Wiener process $\tilde{W}_t$. There is something counter-intuitive about this process. It turns out that both $\tilde{W}_t$ and $W_t$ are standard Wiener processes. Thus, they do not have any drift. But they are related to each other by

$$d\tilde{W}_t = dW_t - X_t dt, \quad (13)$$

which means at least one of them must have nonzero drift. The point is, $\tilde{W}_t$ has zero drift under $\tilde{P}$, whereas $W_t$ has zero drift under $P$. Hence, $\tilde{W}_t$ can be used to represent unpredictable errors in dynamic systems given that we switch the probability measures from $P$ to $\tilde{P}$.

- Finally, $1_A$ is simply a function that has value 1 if $A$ occurs. In fact, we can write:

$$\tilde{P}_T(A) = E^P[1_A \xi_T] = \int_A \xi_T dP \quad (14)$$