8. In many mechanical positioning systems there is flexibility between one part of the system and another. An example is shown in Figure 2.6 where there is flexibility of the solar panels. Figure 2.36 depicts such a situation, where a force \( u \) is applied to the mass \( M \) and another mass \( m \) is connected to it. The coupling between the objects is often modeled by a spring constant \( k \) with a damping coefficient \( b \), although the actual situation is usually much more complicated than this.

(a) Write the equations of motion governing this system.

(b) Find the transfer function between the control input, \( u \), and the output, \( y \).

Solution:

(a) The FBD for the system is

\[
\begin{align*}
\text{m} & \quad \quad \text{b}(\ddot{x} - \dot{y}) \\
\text{k}(x - y) & \quad \quad \text{k}(x - y) \\
\text{b}(\ddot{x} - \dot{y}) & \quad \quad \text{b}(\ddot{x} - \dot{y}) \\
\end{align*}
\]

which results in the equations

\[
\begin{align*}
m\ddot{x} &= -k(x - y) - b(\dot{x} - \dot{y}) \\
M\ddot{y} &= u + k(x - y) + b(\dot{x} - \dot{y})
\end{align*}
\]

or

\[
\begin{align*}
\ddot{x} + \frac{k}{m}x + \frac{b}{m}\dot{x} - \frac{k}{m}y - \frac{b}{m}\dot{y} &= 0 \\
- \frac{k}{M}x - \frac{b}{M}\dot{x}\ddot{y} + \frac{k}{M}y + \frac{b}{M}\ddot{y} &= \frac{1}{M}u
\end{align*}
\]

Figure 2.36: Schematic of a system with flexibility
(b) If we make Laplace Transform of the equations of motion

\[ \begin{align*}
    s^2X + \frac{k}{m}X + \frac{b}{m}sX - \frac{k}{m}Y - \frac{b}{m}sY &= 0 \\
    -\frac{k}{M}X - \frac{b}{M}sX + s^2Y + \frac{k}{M}Y + \frac{b}{M}sY &= \frac{1}{M}U
\end{align*} \]

In matrix form,

\[ \begin{bmatrix} ms^2 + bs + k & -(bs + k) \\
-(bs + k) & Ms^2 + bs + k \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} 0 \\ U \end{bmatrix} \]

From Cramer’s Rule,

\[ Y = \frac{\det \begin{bmatrix} ms^2 + bs + k & 0 \\
-\frac{k}{M}X - \frac{b}{M}sX + s^2Y + \frac{k}{M}Y + \frac{b}{M}sY &= \frac{1}{M}U
\end{align*} \]

\[ Y = \frac{\det \begin{bmatrix} ms^2 + bs + k & -(bs + k) \\
-(bs + k) & Ms^2 + bs + k \end{bmatrix}}{\det \begin{bmatrix} ms^2 + bs + k & -(bs + k) \\
-(bs + k) & Ms^2 + bs + k \end{bmatrix}} = \frac{ms^2 + bs + k}{(ms^2 + bs + k)(Ms^2 + bs + k) - (bs + k)^2}U \]

Finally,

\[ \frac{Y}{U} = \frac{ms^2 + bs + k}{(ms^2 + bs + k)(Ms^2 + bs + k) - (bs + k)^2} = \frac{ms^2 + bs + k}{mMs^4 + (m + M)bs^3 + (M + m)ks^2} \]
16. For a second-order system with transfer function

\[ G(s) = \frac{3}{s^2 + 2s - 3}; \]

determine the following:

(a) The DC gain;

(b) The final value to a step input.

**Solution:**

(a) DC gain \( G(0) = \frac{3}{3} = 1 \)

(b) \( \lim_{t \to \infty} y(t) = ? \)

\[ s^2 + 2s + 3 = 0 \implies s = 1, -3 \]

Since the system has an unstable pole, the Final Value Theorem is not applicable. The output is unbounded.
38. Suppose that unity feedback is to be applied around the listed open-loop systems. Use Routh’s stability criterion to determine whether the resulting closed-loop systems will be stable.

(a) \( KG(s) = \frac{4(s+2)}{s(s^3+2s^2+3s+4)} \)

(b) \( KG(s) = \frac{2(s+4)}{s(s+1)} \)

(c) \( KG(s) = \frac{4(s^3+2s^2+s+1)}{s^3+2s^2+3s+4} \)

Solution:

(a)

\[ 1 + KG = s^4 + 2s^3 + 3s^2 + 8s + 8 = 0 \]

\[
\begin{array}{cccc}
s^4 & 1 & 3 & 8 \\
s^3 & 2 & 8 \\
s^2 & a & b \\
s^1 & c \\
s^0 & d \\
\end{array}
\]

where

\[
a = \frac{2 \times 3 - 8 \times 1}{2} = -1 \quad b = \frac{2 \times 8 - 1 \times 0}{2} = 8 \\
c = \frac{3a - 2b}{a} = \frac{-8 - 16}{-1} = 24 \\
d = b = 8
\]

2 sign changes in first column\( \rightarrow \)2 roots not in LHP\( \rightarrow \)unstable.

(b)

\[ 1 + KG = s^3 + s^2 + 2s + 8 = 0 \]

The Routh’s array is,

\[
\begin{array}{cccc}
s^3 & 1 & 2 \\
s^2 & 1 & 8 \\
s^1 & -6 \\
s^0 & 8 \\
\end{array}
\]

There are two sign changes in the first column of the Routh array. Therefore, there are two roots not in the LHP.
(c) \[ 1 + KG = s^5 + 2s^4 + 3s^3 + 7s^2 + 4s + 4 = 0 \]

\[
\begin{align*}
  s^5 & : 1 \quad 3 \quad 4 \\
  s^4 & : 2 \quad 7 \quad 4 \\
  s^3 & : a_1 \quad a_2 \\
  s^2 & : b_1 \quad b_2 \\
  s^1 & : c_1 \\
  s^0 & : d_1
\end{align*}
\]

where

\[
\begin{align*}
  a_1 &= \frac{6 - 7}{2} = -\frac{1}{2} \quad a_2 = \frac{8 - 4}{2} = 2 \\
  b_1 &= \frac{-7/2 - 4}{-1/2} = 15 \quad b_2 = \frac{-4/2 - 0}{-1/2} = 4 \\
  c_1 &= \frac{30 + 2}{15} = \frac{32}{15} \\
  d_1 &= 4
\end{align*}
\]

2 sign changes in the first column\(\Rightarrow\)2 roots not in the LHP\(\Rightarrow\)unstable.
39. Use Routh’s stability criterion to determine how many roots with positive real parts the following equations have:

(a) \( s^4 + 8s^3 + 32s^2 + 80s + 100 = 0 \).

(b) \( s^5 + 10s^4 + 30s^3 + 80s^2 + 344s + 480 = 0 \).

(c) \( s^4 + 2s^3 + 7s^2 - 2s + 8 = 0 \).

(d) \( s^3 + s^2 + 20s + 78 = 0 \).

(e) \( s^4 + 6s^2 + 25 = 0 \).

Solution:

(a) \( s^4 + 8s^3 + 32s^2 + 80s + 100 = 0 \)

\[
\begin{array}{cccc}
s^4 & : & 1 & 32 & 100 \\
s^3 & : & 8 & 80 \\
s^2 & : & 22 & 100 \\
s^1 & : & 80 - \frac{800}{22} = 43.6 \\
s^0 & : & 100 \\
\end{array}
\]

\( \Rightarrow \) No roots not in the LHP

(b) \( s^5 + 10s^4 + 30s^3 + 80s^2 + 344s + 480 = 0 \)

\[
\begin{array}{cccc}
s^5 & : & 1 & 30 & 344 \\
s^4 & : & 10 & 80 & 480 \\
s^3 & : & 22 & 296 \\
s^2 & : & -545 & 480 \\
s^1 & : & 490 \\
s^0 & : & 480 \\
\end{array}
\]

\( \Rightarrow \) 2 roots not in the LHP.

(c) \( s^4 + 2s^3 + 7s^2 - 2s + 8 = 0 \)

There are roots in the RHP (not all coefficients are >0).
\[ a(s) = s^4 + 6s^2 + 25 = 0 \]

There are two sign changes in the first column of the Routh array. Therefore, there are two roots not in the LHP.

(e)

\[ a(s) = s^4 + 6s^2 + 25 = 0 \]

Two coefficients are missing so there are roots outside the LHP.

Create a new row by \( \frac{da(s)}{ds} \)

\[ a(s) = 0 \implies s^2 = -3 \pm 4j = 5e^{j(\pi \pm 0.92)} \]

\[ s = \sqrt{5}e^{j(\frac{\pi}{4} \pm 0.46) + n\pi j} \quad n = 0, 1 \]
Problem 3.39: s-plane pole locations.
40. Find the range of $K$ for which all the roots of the following polynomial
are in the LHP:

$$s^5 + 5s^4 + 10s^3 + 10s^2 + 5s + K = 0.$$  

Use MATLAB to verify your answer by plotting the roots of the polynomial
in the $s$-plane for various values of $K$.

**Solution:**

$$s^5 + 5s^4 + 10s^3 + 10s^2 + 5s + K = 0$$

$$
\begin{align*}
  s^5 & : 1 & 10 & 5 \\
  s^4 & : 5 & 10 & K \\
  s^3 & :
  & a_1 & a_2 \\
  s^2 & : b_1 & K \\
  s^1 & : c_1 \\
  s^0 & : K
\end{align*}
$$

where

$$
\begin{align*}
  a_1 &= \frac{5 (10) - 1 (10)}{5} = 8 \\
  a_2 &= \frac{5 (5) - 1 (K)}{5} = \frac{25 - K}{8} \\
  b_1 &= \frac{(a_1) (10) - (5) (a_2)}{a_1} = \frac{55 + K}{8} \\
  c_1 &= \frac{(b_1) (a_2) - (a_1) (K)}{b_1} = \frac{- (K^2 + 350K - 1375)}{5 (55 + K)}
\end{align*}
$$

For stability: all terms in first column $> 0$

(1) $b_1 = \frac{55 + K}{8} > 0 \implies K > -55$

(2) $c_1 = \frac{- (K^2 + 350K - 1375)}{5 (55 + K)} > 0, \frac{- (K - 3.88) (K + 354)}{5 (55 + K)} > 0 \implies -55 < K < 3.88$

(3) $d_1 = K > 0$

Combining (1), (2), and (3) $\implies 0 < K < 3.88$. If we plot the roots of
the polynomial for various values of $K$ we obtain the following root locus
plot (see Chapter 5),
Problem 3.40: $s$-plane.