Assignment 9.

This homework is due *Tuesday* Nov 15.

There are total 53 points in this assignment. 42 points is considered 100%. If you go over 42 points, you will get over 100% for this homework (up to 115%) and it will count towards your course grade.

Collaboration is welcome. If you do collaborate, make sure to write/type your own paper *and give credit to your collaborators in your pledge.* Your solutions should contain full proofs. Bare answers will not earn you much.

This assignment covers sections 5.1–5.3 in Bartle–Sherbert.

(1) REMINDER. Recall definition of a continuous function:
Let $A \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$, $c \in A$. We say that f is continuous at c if

$$\forall \varepsilon > 0\ \exists \delta > 0\ \forall x \in A, \quad \text{if } |x - c| < \delta, \text{ then } |f(x) - f(c)| < \varepsilon.$$

Below you can find (erroneous!) "definitions" of a continuous function. In each case describe, exactly which functions are "continuous at c" according to that "definition".

- (a) [4pt] Let $A \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$, $c \in A$. We say that f is "continuous at c" if
$\forall \varepsilon > 0\ \forall \delta > 0\ \forall x \in A, \quad \text{if } |x - c| < \delta, \text{ then } |f(x) - f(c)| < \varepsilon.$
- (b) [4pt] Let $A \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$, $c \in A$. We say that f is "continuous at c" if
$\exists \delta > 0\ \forall \varepsilon > 0\ \forall x \in A, \quad \text{if } |x - c| < \delta, \text{ then } |f(x) - f(c)| < \varepsilon.$
- (c) [6pt] Let $A \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$, $c \in A$. We say that f is "continuous at c" if
$\forall \varepsilon > 0\ \exists \delta > 0\ \exists x \in A, \quad \text{if } |x - c| < \delta, \text{ then } |f(x) - f(c)| < \varepsilon.$

(2) [3pt] (Exercise 5.1.3, Gluing Lemma) Let $a < b < c$. Suppose that f is continuous on $[a, b]$, that g is continuous on $[b, c]$, and that $f(b) = g(b)$. Define h on $[a, c]$ by $h(x) = f(x)$ for $x \in [a, b]$ and $h(x) = g(x)$ for $x \in (b, c]$. Prove that h is continuous on $[a, c]$.

(3)
- (a) [2pt] (Exercise 5.1.5) Let f be defined for all $x \in \mathbb{R}$, $x \neq 2$, by $f(x) = \frac{x^2+x-6}{x-2}$. Can f be defined at $x = 2$ in such a way that f is continuous at this point?
- (b) [2pt] Same question about $g(x) = \frac{x^2+x-7}{x-2}$.

(4)
- (a) [3pt] (Exercise 5.1.12) Suppose $f : \mathbb{R} \to \mathbb{R}$ is continuous on $\mathbb{R}$ and that $f(r) = 0$ for every rational number r. Show that $f(x) = 0$ at every point $x \in \mathbb{R}$.
- (b) [2pt] (Exercise 5.2.8) Let f, g be continuous from $\mathbb{R}$ to $\mathbb{R}$, and suppose that $f(r) = g(r)$ for all rational numbers r. Prove that $f(x) = g(x)$ for all $x \in \mathbb{R}$. (*Hint:* Consider $f - g$.)

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(5) [3pt] (Exercise 5.1.11) (K-Lipschitz functions) Let $K > 0$ and let $f : \mathbb{R} \to \mathbb{R}$ satisfy the condition $|f(x) - f(y)| \leq K|x - y|$ for all $x, y \in \mathbb{R}$. Show that f is continuous on $\mathbb{R}$.

(6) (Exercise 5.2.3) For every $c \in \mathbb{R}$, give an example of functions f and g that are both discontinuous at a point c in $\mathbb{R}$ such that
   (a) [3pt] the sum $f + g$ is continuous at c,
   (b) [3pt] the product fg is continuous at c.
   (*Hint:* Start with $c = 0$, then shift both functions by c.)

(7) [4pt] (Exercise 5.2.5) Let g be defined on $\mathbb{R}$ and by $g(1) = 0$, and $g(x) = 2$ if $x \neq 1$, and let $f(x) = x+1$ for all $x \in \mathbb{R}$. Show that $\lim_{x \to 0} g \circ f \neq (g \circ f)(0)$. Why doesn't this contradict Composition of Continuous Functions Theorem (Theorem 5.2.6)?

(8) (a) [3pt] (Part of exercise 5.3.5) Show that the polynomial $p(x) = x^4 + 7x^3 - 9$ has at least two real roots.
   (b) [4pt] (Exercise 5.3.4) Show that every polynomial of odd degree with real coefficients has at least one real root.

(9) (a) [4pt] (Exercise 5.3.11) Let $I = [a, b]$, let $f : I \to \mathbb{R}$ be continuous on I, and assume that $f(a) < 0$, $f(b) > 0$. Let $W = \{x \in I : \ f(x) < 0\}$, and let $w = \sup W$. Prove that $f(w) = 0$. (This provides an alternate proof of Intermediate Value Theorem.)
   (b) [3pt] Why the same reasoning does not necessarily work if both $f(a) > 0, f(b) > 0$? (That is, find a precise place in the construction above that doesn't go through in such case.)