MA 441, Fall 2013

Assignment 6

This homework is due *Tuesday* Oct 8.

There are total 31 points in this assignment. 27 points is considered 100%. If you go over 27 points, you will get over 100% for this homework and it will count towards your course grade (up to 115%).

Collaboration is welcome. If you do collaborate, make sure to write/type your own paper *and give credit to your collaborators in your pledge*. Your solutions should contain full proofs. Bare answers will not earn you much.

This assignment covers sections 3.2–3.3 in Bartle–Sherbert.

(1) (a) [2pt] (Theorem 3.2.3) Let $X = (x_n)$ and $Y = (y_n)$ be sequences in $\mathbb{R}$ converging to x and y, respectively. Prove that $X - Y$ converges to $x - y$.

(b) [2pt] (Exercise 3.2.3) Show that if X and Y are sequences in $\mathbb{R}$ such that X and $X + Y$ converge, then Y converges.

(c) [2pt] (Exercise 3.2.2b) Give an example of two sequences X, Y in $\mathbb{R}$ such that XY converges, while X and Y do not.

(2) [5pt] Determine the following limits (or establish they do not exist):

(a) $\lim\limits_{n\to\infty} \frac{2n^2-1}{1000n+100000}$,

(b) $\lim\limits_{n\to\infty} \frac{2\sqrt{n^2+1}-10}{1000n+100000}$,

(c) $\lim\limits_{n\to\infty} \frac{2n^2-1}{1000\sqrt[6]{n^5+11}-100000}$.

(3) [2pt] (3.2.12) If $0 < a < b$, determine

$$\lim_{n\to\infty} \left(\frac{a^{n+1} + b^{n+1}}{a^n + b^n} \right)$$

(4) [3pt] (3.2.7) If (b_n) is a bounded sequence and $\lim(a_n) = 0$, show that $\lim(a_n b_n) = 0$. Explain why Theorem 3.2.3 (Arithmetic properties of limit, "$\lim XY = \lim X \cdot \lim Y$") *cannot* be used.

(5) [3pt] (3.2.14) Use the Squeeze Theorem to determine

(a) $\lim\limits_{n\to\infty} (n)^{1/n^2}$,

(b) $\lim\limits_{n\to\infty} (n!)^{1/n^2}$.

You can take for granted that $n^{1/n} \to 1$ as $n \to \infty$. (See Prob. 1e of HW5.)

(6) [2pt] (3.3.2) Let $x_1 > 1$ and $x_{n+1} = 2 - 1/x_n$ for $n \in \mathbb{N}$. Show that (x_n) is bounded and monotone. Find the limit.

(7) [2pt] (3.3.3) Lt $x_1 \geq 2$ and $x_{n+1} = 1 + \sqrt{x_n - 1}$ for $n \in \mathbb{N}$. Show that (x_n) is decreasing and bounded below by 2. Find the limit.

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(8) [2pt] Find a mistake in the following argument:

"Let (x_n) be a sequence given by $x_1 = 1$, $x_{n+1} = 1 - x_n$. In other words, $(x_n) = (1, 0, 1, 0, 1, 0, \ldots)$. Show that $\lim(x_n) = 0.5$. Indeed, let $\lim(x_n) = x$. Apply limit to both sides of equality $x_{n+1} = 1 - x_n$:

$$\lim(x_{n+1}) = \lim(1 - x_n)$$
$$\lim(x_{n+1}) = 1 - \lim(x_n)$$
$$x = 1 - x,$$

so $x = 0.5$"

(9) [2pt] (3.3.10) Establish convergence or divergence of the sequence (y_n), where

$$y_n = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} \qquad \text{for } n \in \mathbb{N}.$$

(10) (a) [2pt] (3.3.11) Let $x_n = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2}$, $n \in \mathbb{N}$. Prove that that (x_n) converges. (*Hint*: for $k \geq 2$, $\frac{1}{k^2} \leq \frac{1}{k(k-1)} = \frac{1}{k-1} - \frac{1}{k}$.)

(b) [2pt] Let K be a natural number $K \geq 2$. Let $y_n = \frac{1}{1^K} + \frac{1}{2^K} + \frac{1}{3^K} + \cdots + \frac{1}{n^K}$, $n \in \mathbb{N}$. Prove that that (y_n) converges. (*Hint*: compare y_n to x_n.)