Assignment 9.

This homework is due Thursday, October 31.

Collaboration is welcome. If you do collaborate, make sure to write/type your own paper and credit your collaborators. Your solutions should contain full proofs. Bare answers will not earn you much. Extra problems (if there are any) are due December 6.

1. Quick reminder

(B) For a bounded function on a set $E$ of finite measure, Lebesgue integral $\int_E f$ is defined as the common value of $\sup\{\int_E \varphi \mid \varphi \leq f, \varphi \text{ simple}\}$ and $\inf\{\int_E \psi \mid \psi \geq f, \psi \text{ simple}\}$, if the latter two are equal (which is guaranteed if $f$ is measurable).

(P) Further, for an arbitrary nonnegative measurable function $f : E \to \mathbb{R} \cup \{\pm \infty\}$, define its Lebesgue integral by

$$\int_E f = \sup \left\{ \int_E h \biggm| h \text{ bounded, measurable, of finite support and } 0 \leq h \leq f \text{ on } E \right\}$$

Both integrals defined above in (B) and (P) are linear, monotone and domain additive. Moreover, the following convergence theorem holds.

**The Bounded Convergence Theorem.** Let $\{f_n\}$ be a sequence of measurable functions on a set of finite measure $E$; let $f_n g$ be uniformly bounded on $E$. If $f_n \to f$ pointwise on $E$, then

$$\lim_{n \to \infty} \int_E f_n = \int_E f.$$ 

2. Exercises

(1) (4.1.6+) Prove that a continuous function on a closed interval $[a, b]$ is Riemann integrable. (*Hint:* Use uniform continuity to put an upper bound on the difference between upper and lower Darboux sums.)

(2) (4.2.9, 3.17) Let $E$ have measure zero. Show that if $f$ is a function on $E$, then $f$ is measurable and $\int_E f = 0$

(a) for the definition (B) (assuming $f$ is bounded),
(b) for the definition (P) (assuming nonnegative $f : E \to \mathbb{R} \cup \{\pm \infty\}$, including the case $f = \infty$ everywhere on $E$).

(3) (4.2.10+) Let $f$ be a measurable function on a set $E$. For a measurable subset $A$ of $E$, show that $\int_A f = \int_E \chi_A f$

(a) for the definition (B) (assuming $f$ is bounded and $E$ is of finite measure),
(b) for the definition (P).

(4) (4.2.13) Show that the Bounded convergence theorem fails if we drop

(a) the assumption that the sequence $\{f_n\}$ is uniformly bounded,
(b) the assumption $m(E) < \infty$. 

— see next page —
(5) Let $f$ be a semisimple function, i.e. a function of the form $f = \sum_{n=1}^{\infty} \lambda_n \chi_{E_n}$ for some measurable sets $E_n$ and real numbers $\lambda_n$. Assume additionally that $f$ is bounded and of finite support. Prove that $\int_R f = \sum_{n=1}^{\infty} \lambda_n m(E_n)$. (Hint: Use the Bounded convergence theorem.)

(6) (4.3.19) For a number $\alpha \in \mathbb{R}$, define $f(x) = x^{\alpha}$ for $0 < x \leq 1$ and $f(0) = 0$. Compute $\int_{[0,1]} f$. (Hint: In the bounded case, use connection to the Riemann integral. For the unbounded case, assume $h$ in (P) is bounded by a number $M$. Using $h \leq f$, compute the largest possible value of $\int_{[0,1]} h$.)

(7) (4.3.18) Let $f$ be a bounded nonnegative function on a set $E$ of finite measure. Show that the definitions (B) and (P) agree on $f$, i.e. they both give the same value of $\int_E f$. 