Assignment 7.

Cauchy Theorem.

This assignment is due Wednesday, March 6. Collaboration is welcome. If you do collaborate, make sure to write/type your own paper.

(1) Suppose that $f(z)$ is analytic in the closed domain $0 \le \arg z \le \alpha$ (where $0 \le \alpha \le 2\pi$), and $\lim_{z\to\infty} zf(z) = 0$. Prove that if the integral
$$J_1 = \int_0^\infty f(x)dx$$
exists, then so does the integral
$$J_2 = \int_L f(z)dz,$$
where L is the ray $z = re^{i\alpha}$, $0 \le r < \infty$. Moreover, show that $J_1 = J_2$. (Hint: use Cauchy's theorem and Problem 5 of HW6.)

(2) Prove that
$$\int_L \frac{dz}{z-a} = \pm 2\pi i$$
if L is any closed rectifiable simple curve whose interior contains the point a (*plus* if L is traversed counterclockwise, *minus* if clockwise). (*Hint:* Use Cauchy theorem for a system of contours.)

(3) Prove that
$$\int_L \frac{dz}{z^2+1} = 0$$
if L is any closed rectifiable simple curve in the outside of closed unit disc, i.e. L is contained in the region $|z| > 1$.
Show that the equality is in general false for arbitrary closed rectifiable simple curves that miss zeros of $z^2 + 1$.

(4) Evaluate the integral
$$\int_{|z-i|=R} \frac{z^4+z^2+1}{z(z^2+1)}$$
as a function of $R > 0$. You may omit values of R for which the denominator turns to 0. (Hint: $\frac{z^4+z^2+1}{z(z^2+1)} = z + \frac{1}{z} - \frac{1}{2}(\frac{1}{z+i} + \frac{1}{z-i})$.)

(5) Let $p(z) = (z-z_1)(z-z_2)\cdots(z-z_n)$, $z_k \ne z_j$ when $k \ne j$. Let L be a simple closed rectifiable curve that does not pass through any of the points $z_1, \dots, z_n$. How many distinct values can $\int_L \frac{dz}{p(z)}$ have, at most?

(6) Compute the integral
$$\int_{-\infty}^{\infty} e^{-x^2}\cos(2bx)dx, \qquad b \in \mathbb{R}, b > 0.$$
(*Hint:* Integrate e^{-z^2} along the path shown in the figure. You can take for granted that $\int_{-\infty}^{\infty} e^{-t^2}dt = \sqrt{\pi}$. Integrals along the vertical sides should go to zero.)