Assignment 9–10.

Taylor Series. Uniqueness Theorem.

This assignment is due Wednesday, April 3. Collaboration is welcome. If you do collaborate, make sure to write/type your own paper.

In this homework, you can take for granted basic inequalities involving limsup and arithmetic operations. You also can take for granted that $\sqrt[n]{\frac{n!}{n^n}} \to \frac{1}{e}$ as $n \to \infty$.

(1) Find the radius of convergence of each of the following power series.

(a) $\sum_{n=1}^{\infty} n^k z^n$ $(k = 0, 1, 2, \ldots)$; (b) $\sum_{n=1}^{\infty} n^n z^n$; (c) $\sum_{n=1}^{\infty} 2^n z^n$;

(d) $\sum_{n=1}^{\infty} (3 + (-1)^n)^n z^n$; (e) $\sum_{n=1}^{\infty} (\cos in) z^n$; (f) $\sum_{n=1}^{\infty} \frac{n^k}{n!} z^n$;

(g) $\sum_{n=1}^{\infty} \frac{n!}{n^n} z^n$; (h) $\sum_{n=1}^{\infty} (n + a^n) z^n$ $(a \in \mathbb{C})$; (i) $\sum_{n=1}^{\infty} z^{n^2}$;

(j) $\sum_{n=1}^{\infty} 2^n z^{n!}$.

(2) Given that the radius of convergence of the power series $\sum_{n=1}^{\infty} c_n z^n$ is R $(0 < R < \infty)$, what is the radius of convergence of the following series?

(a) $\sum_{n=1}^{\infty} n^k c_n z^n$ $(k = 0, 1, 2, \ldots)$; (b) $\sum_{n=1}^{\infty} (2^n - 1) c_n z^n$; (c) $\sum_{n=1}^{\infty} \frac{c_n}{n!} z^n$;

(d) $\sum_{n=1}^{\infty} c_n^k z^n$ $(k = 0, 1, 2, \ldots)$; (e) $\sum_{n=1}^{\infty} c_n z^{n^2}$.

(3) Under assumptions of Problem 2, what can you say about the radii of convergence of (a) $\sum_{n=1}^{\infty} c_{n^2} z^{n^2}$; (b) $\sum_{n=1}^{\infty} c_{n^2} z^n$?

(4) Let the radii of convergence of the power series $\sum_{n=1}^{\infty} a_n z^n$ and $\sum_{n=1}^{\infty} b_n z^n$ equal R_1 and R_2, respectively. Let R be the radius of convergence of

(a) $\sum_{n=1}^{\infty} (a_n + b_n) z^n$. Prove that $R \geq \min(R_1, R_2)$.

(b) $\sum_{n=1}^{\infty} a_n b_n z^n$. Prove that $R \geq R_1 R_2$.

(5) (a) Give an example of two power series $\sum_{n=1}^{\infty} a_n z^n$ and $\sum_{n=1}^{\infty} b_n z^n$ with the same finite radius of convergence $R_1 = R_2$, such that the radius of convergence of $\sum_{n=1}^{\infty} (a_n + b_n) z^n$ is infinite.

(b) If $R_1 < R_2$, can the radius of convergence of $\sum_{n=1}^{\infty} (a_n + b_n) z^n$ be strictly greater than R_1?

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(6) Give an alternative prove of Liouville's theorem, based on the use of Cauchy's integral formula.
(*Hint:* Consider the integral
$$\int_{|z|=R} \frac{f(z)}{(z-a)(z-b)} dz,$$
where $a \neq b$, $|a| < R$, $|b| < R$, and take $R \to \infty$.)

(7) Prove that following generalization of Liouville's theorem.
If $f(z)$ is an entire function, and if the function
$$M(\rho) = \max_{|z|=\rho} |f(z)|$$
satisfies the inequality
$$M(\rho) \leq M\rho^k$$
for some fixed positive M and a fixed integer k, then $f(z)$ is a polynomial of degree no higher than k.
(*Hint:* Prove that the k-th derivative of f is constant.)

(8) Find Taylor series at $z = 0$ and its radius of convergence of the following functions.
(a) $\sin^2 z$; (b) $\frac{1}{az+b}$ $(a, b \in \mathbb{C}, b \neq 0)$; (c) $\frac{1}{z^2-5z+6}$; (d) $\int_0^z e^{\zeta^2} d\zeta$; (e) $\int_0^z \frac{\sin\zeta}{\zeta} d\zeta$.
(*Hint:* Here, you can take for granted the following not entirely obvious fact. If Φ and f are analytic, then Taylor series for $\Phi(f)$ can be obtained by formal substitution of Taylor series of f into the Taylor series of Φ. [You really need it only in (8d), though.])

(9) Taking Cauchy integral theorem, Taylor decomposition theorem and analyticity of sum of power series for granted, prove Cauchy integral formula by integrating term by term. (*Hint:* In particular, don't forget to explain, *why* you can integrate term by term.)
Comment. Taking Taylor decomposition theorem and analyticity of sum of power series for granted, it is also possible to prove Cauchy integral theorem by a direct computation, similar to Problem 1 of HW5.

(10) Does there exist a function that is analytic on a neighborhood of $z = 0$ and takes the following values at $z = 1/n$ $(n = 1, 2, \ldots)$:
(a) $0, 1, 0, 1, 0, 1, 0, 1, \ldots$;
(b) $0, \frac{1}{2}, 0, \frac{1}{4}, 0, \frac{1}{6}, \ldots, 0, \frac{1}{2k}, \ldots$;
(c) $\frac{1}{2}, \frac{1}{2}, \frac{1}{4}, \frac{1}{4}, \frac{1}{6}, \frac{1}{6}, \ldots, \frac{1}{2k}, \frac{1}{2k}, \ldots$;
(d) $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \frac{6}{7}, \ldots, \frac{n}{n+1}, \ldots$?
(*Hint:* Use the interior uniqueness theorem.)

(11) Does there exist a function that is analytic on a neighborhood of $z = 0$ and satisfies the following condition for every positive n:
(a) $f(1/n) = f(-1/n) = 1/n^2$ (*Hint:* Yes);
(b) $f(1/n) = f(-1/n) = 1/n^3$ (*Hint:* No)?