Assignment 13.

Residues and stuff

This assignment is due Wednesday, April 29. Collaboration is welcome. If you do collaborate, make sure to write/type your own paper.

(1) Compute
  (a) $\int_{-\infty}^{\infty} \frac{dx}{x^2+p^2}$, for $p > 0$,
  (b) $\int_{-\infty}^{\infty} \frac{dx}{(x^2+p^2)^2}$, for $p > 0$.

(2) Find the following residues:
  (a) $\operatorname{res}_0 1/e^z$.
  (b) Find residues of $\frac{z^2}{(z+1)^2(z-4)(z+3)}$ at all its poles.
  (c) Is residue of $\sin(1/z)$ defined at $z = 0$? If yes, find the residue; if not, explain why.
  (d) Same question about $\frac{1}{\sin(1/z)}$.

(3) (a) Find $\operatorname{res}_a \frac{\varphi(z)}{(z-a)^n}$, where φ is a given function analytic at a, $\varphi(a) \neq 0$, and n is a positive integer.
  (b) Suppose a is a simple pole of f, and let $\operatorname{res}_a f = A$. Find $\operatorname{res}_a (\varphi f)$, where $\varphi(z)$ is analytic at a.

(4) In this problem we compute the sum $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots$.
  (a) Find first two nonzero terms of the Laurent series for $\cot z$ at 0. (*Hint:* For example, you can find first few terms of the Taylor series at 0 of analytic function $z \cot z$ by differentiation; or you can straightforwardly divide cos by sin.)
  (b) Find all isolated singular points of $\cot(\pi z)$. Find the corresponding residues. (*Hint:* The points are $z = k$, $k \in \mathbb{Z}$.)
  (c) Consider the function $\frac{\cot(\pi z)}{z^2}$. Find its residues at all its isolated singular points. (*Hint:* At $k \neq 0$, use Problem 3b. Treat $z = 0$ separately using 4a.)
  (d) Let γ_n be the circle $R_n e^{it}$, $0 \leq t \leq 2\pi$, where $R_n = n + \frac{1}{2}$. Find the integral
  $$\int_{\gamma_n} \frac{\cot(\pi z)}{z^2} dz$$
  using the residue theorem.
  (e) Show that for a fixed $t \neq m\pi$, $|\cot(\pi R_n e^{it})| \to 1$ as $n \to \infty$.
  Show that for a fixed $t = m\pi$, $|\cot(\pi R_n e^{it})| \to 0$ as $n \to \infty$.
  (*Hint:* For example, you can use formulas $|\sin z|^2 = \sinh^2 y + \sin^2 x$, $|\cos z|^2 = \cosh^2 y - \sin^2 x$ that we got in Problem 5 of HW4.)

— see next page —

(f) *(This item is optional, i.e. not included in the denominator of the grade. You can take it for granted in subsequent argument.)* Conclude that $|\cot(\pi z)|$ is eventually bounded by 2 on circles γ_n. (*Hint:* This is not hard, just a bit technical.)

(g) Conclude that

$$\int_{\gamma_n} \frac{\cot(\pi z)}{z^2} dz \ \to 0 \quad \text{as } n \to \infty.$$

(*Hint:* Use $|\int f dz| \leq ML$.)

(h) Put together 4d and 4g to compute $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots$ (*Hint:* If your answer is not $\pi^2/6$, something is wrong.)

(5) Similar to the Problem 4, find the sum

$$\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \ldots.$$

You can take for granted that at 0, $\cot z = \frac{c_{-1}}{z} + c_1 z - z^3/45 + \ldots$, where c_{-1}, c_1 were found in 4a.
(*Hint:* If your answer is not $\pi^4/90$, something is wrong.)

COMMENT. The same technique can be used to find $\sum_{n=-\infty}^{\infty} R(n)$, where $R(n)$ is an arbitrary rational function with the denominator $Q(z)$ at least 2 degrees higher than the numerator, and $Q(n) \neq 0$ at $n \in \mathbb{Z}$. The argument is even a bit easier because under such constraints, because there is no need to deal with the annoying residue at 0 separately.

In the case $Q(n) = 0$ for some n (e.g. $R(z) = 1/z^2$ or $1/z^4$ as above), one can find the same sum, omitting the infinite terms (as we actually did in the Problems 4 and 5).

QUESTION. Why doesn't this help find $\frac{1}{1^3} + \frac{1}{2^3} + \frac{1}{3^3} + \ldots$?