Assignment 2.

Convergence in $\mathbb{C}$. The extended complex plane. Map $1/z$.

This assignment is due Wednesday, Feb 3. Collaboration is welcome. If you do collaborate, make sure to write/type your own paper.

(1) Suppose a sequence (z_n) in $\mathbb{C}$ converges to $z \in \mathbb{C}$ as $n \to \infty$.
  (a) Is it true that $|z_n| \to |z|$, $\operatorname{Re} z_n \to \operatorname{Re} z$, $\operatorname{Im} z_n \to \operatorname{Im} z$ $(n \to \infty)$?
  (b) Is it true that $\arg z_n \to \arg z$ $(n \to \infty)$? (*Hint:* Consider $z = 0$.)
  (c) Is it true that $\arg z_n \to \arg z$ $(n \to \infty)$, provided $z \neq 0$? (*Hint:* Still no.)
  (d) Provided $z \neq 0$, is it possible to choose a value φ_n of $\operatorname{Arg} z_n$ for each n so that $\varphi_n \to \arg z$ $(n \to \infty)$?

(2) Suppose the sequence (z_n) in $\mathbb{C}$ converges to infinity as $n \to \infty$. What does this imply about $|z_n|$, $\operatorname{Re} z_n$, $\operatorname{Im} z_n$, $\operatorname{Arg} z_n$?

(3) Assuming arithmetic operations on $\overline{\mathbb{C}}$ are defined via arithmetic operations on the corresponding sequences, give examples showing why $\infty - \infty$, $0 \cdot \infty$, ∞/∞, $0/0$ are meaningless.

(4) Find and sketch the images of the following curves under the transformation $w = 1/z$ (*Hint:* It is probably more convenient to use complex equations for the families below):
  (a) The family of circles $x^2 + y^2 = ax$ $(a \in \mathbb{R})$. Remark: compare to Prob. 9 of HW1. (*Hint:* Rewrite the equation in terms of $z, \bar{z}$. Then plug in $z = 1/w$.)
  (a′) The family of vertical lines $\operatorname{Re} z = a$ $(a \in \mathbb{R})$. (*Hint:* This item is named (a′) for a reason.)
  (b) The family of circles $x^2 + y^2 = by$ $(b \in \mathbb{R})$. Remark: compare to Prob. 9 of HW1.
  (b′) The family of horizontal lines $\operatorname{Im} z = b$ $(b \in \mathbb{R})$.
  (c) The family of parallel lines $y = x + b$ $(b \in \mathbb{R})$.
  (d) The family of lines $y = kx$ passing through the origin $(k \in \mathbb{R})$.

(5) Find and sketch the images of the following regions on $\overline{\mathbb{C}}$ under the transformation $w = 1/z$.
  (a) The disc $x^2+y^2 < 4$. (*Hint:* First figure out where the circle $x^2+y^2 = 4$ goes. Of the two regions bounded by the image of the circle, the inside and the outside, one is the answer. To find out which, it suffices to either think hard, or to test where a single (specific) point in $x^2+y^2 < 4$ goes. Follow a similar procedure in all the other items.)
  (b) The quadrant $x > 0,\ y > 0$.
  (c) The strip $0 < x < 1$.
  (d) The half-plane $x > 10$.
  (e) The outside of a circle $x^2+(y-1)^2 = 1$ (i.e. the region $x^2+(y-1)^2 > 1$).
  (f) The outside of a circle $x^2+(y-2)^2 = 1$ (i.e. the region $x^2+(y-2)^2 > 1$).
  (g) The square $-1 \leq x \leq 1$, $-1 \leq y \leq 1$.
  (h) The upper half-disc $|z| < 1$, $\operatorname{Im} z > 0$.