Assignment 5.

Basics of complex integral. Random stuff

This assignment is due Wednesday, Feb 24. Collaboration is welcome. If you do collaborate, make sure to write/type your own paper.

1. Integral

(1) Compute the integral $\int_\gamma \frac{dz}{z}$, where $\gamma : [0, 2\pi] \to \mathbb{C}$ is a circle $\gamma(t) = re^{it}$, as follows. Let $\dot{\mathcal{P}}$ be a partition where $\gamma(t_k)$'s divide circle into equal arcs, $t_k = 2k\pi/n$ $(k = 0, 1 \dots, n)$; $\gamma(\tau_k)$ are the midpoints of corresponding arcs, $\tau_k = (2k-1)\pi/n$ $(k = 1, \dots, n)$. Find $S(f, \dot{\mathcal{P}})$ and its limit as $|\dot{\mathcal{P}}| \to 0$.
(*Hint:* Stick to exponentials until the last moment.)
Comment. The function above is continuous on the path of integration, so we know from calculus that the integral exists. Therefore, the limit you found above actually equals to the value of the integral. In particular, if your answer is not $2\pi i$, something is wrong.

(2) Evaluate the integrals $J_1 = \int_L x\,dz$ and $J_2 = \int_L y\,dz$ along the following curves. (*Hint:* Parametrize and use $\int_\gamma f(z)dz = \int_a^b f(\gamma(t))\gamma'(t)dt$.)
   (a) The line segment joining points $z = 0$ and $z = 2 + i$,
   (b) The semicircle $|z| = 1$, $\text{Im}z \geq 0$, with initial point $z = 1$,
   (c) The circle $|z - a| = R$.

(3) Evaluate the integrals $\int_L |z|\,dz$ along the following curves. (*Hint:* Use parametrization.)
   (a) The circle $|z| = R$.
   (b) The semicircle $|z| = 1$, $-\pi/2 \leq \arg z \leq \pi/2$, with initial point $z = -i$.

(4) Show that
$$\lim_{r\to 0} \int_0^{2\pi} f(re^{i\varphi})d\varphi = 2\pi f(0),$$
if f is continuous on a neighborhood of $z = 0$.
(*Hint:* Write $\int_0^{2\pi} f(re^{i\varphi})d\varphi = \int_0^{2\pi} f(0)d\varphi + \int_0^{2\pi}(f(re^{i\varphi}) - f(0))d\varphi$. Use Ml-inequality to show that the latter integral $\to 0$.)

(5) Show that
$$\lim_{r\to 0} \int_{|z-a|=r} \frac{f(z)}{z-a}dz = 2\pi i f(a),$$
if f is continuous on a neighborhood of $z = a$. (*Hint:* Parametrize and use the previous problem.)

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2. Stuff

Problems below don't necessarily have anything to do with the integral. Rather, they highlight some differences between real and complex numbers.

(6) Prove that there does not exist a subset P of $\mathbb{C}$ with the following properties:
- For every $z \in \mathbb{C}$, exactly one of the following holds: $z \in P$, or $-z \in P$, or $z = 0$.
- For every $z, w \in P$, the sum is also in P: $z + w \in P$.
- For every $z, w \in P$, the product is also in P: $zw \in P$.

Comment. This is mostly a repetition of what I said in class about order, that is an explanation that $>$ and $<$ cannot be defined on entire $\mathbb{C}$ consistently with arithmetic operations. If they were, then the set $\{z > 0\}$ would work as such P. (P stands for *positive*, by the way.)

(7) Give a counterexample to Mean Value Theorem for complex numbers. For instance, you can find a function $f : \mathbb{C} \to \mathbb{C}$ s.t. $f(z_1) = f(z_2)$ for some $z_1 \neq z_2$, but f' is never 0. (*Hint:* What functions with nowhere zero derivative do you know?)

(8) Prove that if two functions $f, g : G \to \mathbb{C}$ defined on a domain $G \subseteq \mathbb{C}$ are differentiable on G and $f' = g'$, then $f = g + C$ for some constant C. (*Hint:* Note that in real analysis, this is proved using the mean value theorem which fails for complex numbers as seen above. Here, instead, use expression of f' through partial derivatives.)

(9) Compute $\int_\gamma e^{2\pi i z} dz$, where γ is the interval $[0, 1]$ of real line. Observe that the Mean Value Theorem for integral fails, too.