ME345 – Modeling and Simulation **NAME:** KEY
Fall 2005
December 9, 2005

NOTE: For the final you may use up to five (5) 8 1/2'' by 11'' sheets of notes that you have prepared in reviewing for this test. You MAY NOT use review sheets that have been prepared by others in the class.

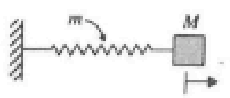
25

PROBLEM 1.
An often used formula in the area of vibrations is that the natural frequency ω_{nat} of a spring-mass system such as that shown in Figure 1 is given by

$$\omega_{nat} = \sqrt{\frac{k}{M}} \qquad \text{(Equation 1)}$$

where k is the stiffness of the spring and M is value of the mass at the end of the spring.

Equivalent masses



Mass (M) attached at end of spring of mass m $\qquad m_{eq} = M + \frac{m}{3}$

Figure 1. One degree of freedom spring-mass system.

In actuality, such an expression only holds when the mass of the spring negligible (what some call a *massless spring*). If the mass of the spring is NOT negligible, the natural frequency of the system can be written in terms of the *equivalent mass* such that

$$\omega_{nat} = \sqrt{\frac{k}{M_{eq}}} \text{ , where } M_{eq} = M + m/3 \qquad \text{(Equation 2)}$$

is the equivalent mass of the system, M is the mass at the end of the spring and m is the mass of the spring.

6

1. For any spring-mass system, how will accounting for the mass of the spring affect the derived natural frequency of the system?

$M_{eq} > M \Rightarrow$ natural frequency will be smaller

6

2. Show that the "limit case" of Equation 2 provides the expression shown in Equation 1.

as $m << M$, $M + \frac{m}{3} \rightarrow M$.

results are the same.

13

3. What is the ratio of (M/m) such that the natural frequencies predicted in Equations (1) and (2) agree within 1% for the same spring constant of the system. Prove that your result is correct.

$$\frac{\sqrt{\frac{K}{M}}}{\sqrt{\frac{K}{M+\frac{m}{3}}}} \leq 1.01$$

$$\frac{M + m/3}{M} \leq 1.0201$$

a 16.583

$$\frac{m}{3} \leq .0201 M \quad \Rightarrow \quad \boxed{\frac{m}{M} \leq .0603}$$

Test Case: $M = 1$, $m = .05$ $\left(\frac{m}{M} = .05\right)$, $k = 1$

EQ 1: $\sqrt{\frac{1}{1}} = 1$

EQ 2: $\sqrt{\frac{1}{1+\frac{.05}{3}}} = \sqrt{\frac{1}{1.0167}} = .9918$ ✓

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PROBLEM 2.

In the design of a heat sink the geometry of the symmetric fin shown in Figure 2 is to be optimized. Specifically, you are given the task to minimize the volume of the fin given the constraint that the "exposed" surface area (i.e. not including the bottom surface area) of the fin must be $\geq$ 20 mm^3. The range of variables for the fin top (d1), bottom (d2), and height (h) are given in Figure 3.

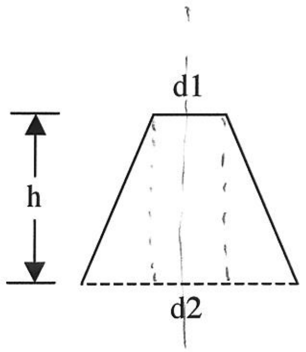
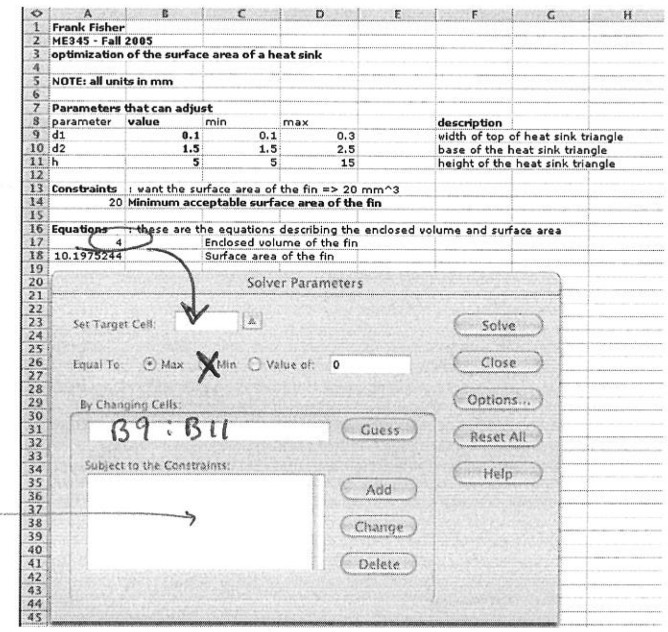


Figure 2. Fin design for a heat sink.



$0.1 \leq d_1 \leq 0.3$

$1.5 \leq d_2 \leq 2.5$

$5 \leq h \leq 15$

A18 $\geq$ A14 (SA constraint)

Figure 3. Excel optimization study for the fin design.

10

1. Identify and derive any equations necessary to solve this optimization problem using the Excel solver tool. (Hint: recall that $c = \sqrt{a^2 + b^2}$)

SA: $= 2\left[\sqrt{\left(\frac{d_2}{2} - \frac{d_1}{2}\right)^2 + h^2}\right] + d_1$ (check = 10.1975 ✓)

$V = 2\left[\left(\frac{1}{2}\right)\left(\frac{d_2}{2} - \frac{d_1}{2}\right) h\right] + d_1 h$ (check = 4 ✓)

2. Explain in words (or alternatively using Figure 3) how to solve this multivariate problem using the Excel solver tool.

15

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PROBLEM 3.

In class we have discussed the element stiffnesses for both a spring and a rod element, which are given as:

Spring element stiffness matrix: $$k\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} d_1 \\ d_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix}$$

Rod element stiffness matrix: $$\frac{AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} d_1 \\ d_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix}$$

We will now consider the torsional element (also referred to as a torsional spring) shown in Figure 4, which has the element spring constant given below

$$\frac{JG}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix}$$

where J is the polar moment of inertia, G is the shear modulus, and L is the element length. Note that the torsional spring stiffness relates *nodal angle of twist* θ to the *nodal applied torque T*.

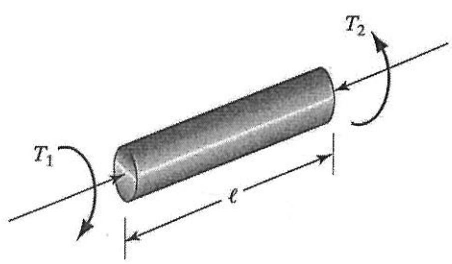


Figure 4. Torsional spring element.

Hint: recall that for a solid cylindrical rod the polar moment of inertia is given as $J = \frac{1}{2}\pi r^4$ where r is the radius of the rod. *Be careful with units!*

For the problem shown in For element 1: r1 = 2 inches, L1 = 2 ft, G1 = 3 * 10^6 lb/in^2

For element 2: r2 = 1 inch, L2 = 1 ft, G2 = 3 * 10^6 lb/in^2

Figure 5a (clearly show and label all work):

1. assemble the *global* stiffness matrix.

2. solve for the angle of twist at node 2.

3. justify your analysis by considering the angle of twist for a torque of 200 lb*ft applied on only element 1. (i.e. see Figure 5b)

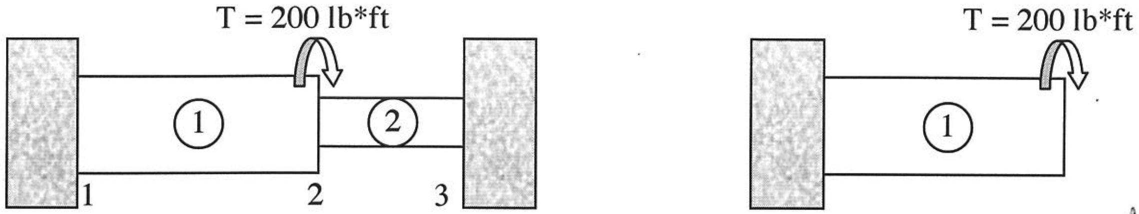
Part 3 $\rightarrow$ Comparison.

$$\theta = \frac{TL}{JG} = \frac{(200\ lb\cdot ft)\left(\frac{12\ in}{1ft}\right)(2ft)\left(\frac{12\ in}{1ft}\right)}{(8\pi\ in^4)(3\times10^6\ lb/in^2)}$$

8

$$\boxed{\theta = 7.64\times10^{-4}\ rad}$$

this is more because less resistance to twist in this case!



For element 1: r1 = 2 inches, L1 = 2 ft, G1 = 3 * 10^6 lb/in^2

For element 2: r2 = 1 inch, L2 = 1 ft, G2 = 3 * 10^6 lb/in^2

Figure 5a. (left) Torsional problem via the finite element method. (right) Comparison problem.

Ele 1: $\frac{JG}{L} = \frac{(8\pi\ in^4)(3\times10^6\ \frac{lb}{in^2})}{2ft\times\frac{12\ in}{1ft}} = \pi\times10^6\ lb\cdot in\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}$

$J_1 = \frac{1}{2}\pi(2)^4 = 8\pi\ in^4$

Ele 2: $\frac{JG}{L} = \frac{(\frac{\pi}{2}\ in^4)(3\times10^6\ lb/in^2)}{1ft\times\frac{12\ in}{1ft}} = \frac{\pi}{8}\times10^6\ lb\cdot in\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}$

$J_2 = \frac{1}{2}\pi(1)^4 = \frac{\pi}{2}\ in^4$

9

GLOBAL STIFFNESS MATRIX

$$\begin{Bmatrix}T_1\\ T_2\\ T_3\end{Bmatrix} = \pi\times10^6\ lb\cdot in\begin{bmatrix}1 & -1 & 0\\ -1 & 1+\frac{1}{8} & -\frac{1}{8}\\ 0 & -\frac{1}{8} & -\frac{1}{8}\end{bmatrix}\begin{Bmatrix}\theta_1\\ \theta_2\\ \theta_3\end{Bmatrix}$$

$$T_2 = \frac{9}{8}\pi\times10^6\ \theta_2$$

8

$$200\ lb\cdot ft\times\frac{12\ in}{1ft} = \left(\frac{9}{8}\pi\times10^6\right)(\theta_2) \Rightarrow \boxed{\theta_2 = 6.79\times10^{-4}\ rad}$$

angle of twist at Node 2

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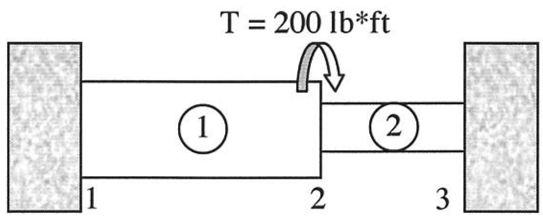
PROBLEM 4. The Matlab code below was used in class to implement the finite element method to solve one dimensional problems with rod elements. Edit the appropriate lines of code to solve the previous problem where 1D torsional elements are used. Be sure to use consistent units!



$$\frac{AE}{L} \leftrightarrow \frac{JG}{L}$$

For element 1: r1 = 2 inches, L1 = 2 ft, G1 = 3 * 10^6 lb/in^2

For element 2: r2 = 1 inch, L2 = 1 ft, G2 = 3 * 10^6 lb/in^2

```
% One dimensional finite element program
% Frank Fisher, Stevens Institute of Technology
% Modeling 345, Fall 2005
%
% INPUT DATA
% this data automatically describes a line of 2-node elements
% with support conditions at both ends and a uniform load; the
% number of elements can be varied by changing numele
clear;
clc;
disp('This code was originally written for a 1D rod element.');
disp('Stepped rod problem');
numele=3;2
numnod=numele+1;
x=[0,5,10,15]; [0,24,36]
node= [1:numele; 2:numele+1];
area=[1,0.5,0.25];    J=[8π, π]
young=[200,100,20];  G=3x10^6 [1  1]
% support conditions, ifix(i)=1 if node i is fixed, else zero
ifix=[1,zeros(1,numele)];
ifix(numnod)=1;
% applied forces
force=[0,20,30,50]; Torque=[0,200x12,0]
% DO NOT EDIT BELOW THIS LINE
{rest of code follows}
```