EXAMPLE 8: An Electrical System (Mechanical-Electrical Analogy)

A completely analogous procedure can be used to find the state equations of electrical systems (and, ultimately, electro-mechanical systems with transformers between the electrical and mechanical domains). We will outline this procedure in the Electrical Domain below with an example.

Note: As you may have figured by now, the ultimate goal of this process is to develop one or more differential equation that describe the behavior of the system as a function of time. Once this equation(s) has been determined, it is simply a matter of choosing one's favorite approach to solve the differential equations (these solution techniques are covered in depth in Differential Equations).

To start, it will be useful to outline the analogy between the Mechanical and Electrical domains. As far as the representative elements are concerned, in the Mechanical domain we had **springs, dampers, and masses**, whereas in the Electrical domain we will consider **capacitors, resistors, and inductors**. The analogy is explored more in the table below:

	'Forcing' Term	**'Flow' Term**	**Time derivatives**
Mechanical	Force	Velocity (displacement)	Displacement -> Velocity -> Acceleration
Electrical	Voltage	Current (Charge)	Charge -> Current -> di/dt

Analogy between Mechanical and Electrical Terms

	Mechanical Domain		**Electrical Domain**	
Store energy	Spring (k)	$f_s = kx_s$	Capacitor (C)	$V_c = \dfrac{1}{C} q_C$
Dissipate energy	Damper (b)	$f_d = bv_d = b\dot{x}_d$	Resistor (R)	$V_R = Ri_R = R\dot{q}_R$
'Inertial' behavior	Mass (m)	$f_m = ma_m = m\ddot{x}_m$	Inductor (L)	$V_I = L\dfrac{di}{dt} = L\ddot{q}_\iota$

Relationship between Constitutive Elements in Mechanical & Electrical Domains

If we reconsider the **Five Step to Solving a State Equation** that were discussed before (and to now only demonstrated for a Mechanical System), from the Table above we have the constitutive relationships for the elements comprising our electrical system (**Step 1**).

Step 2 in the (mechanical system) process was referred to as Geometric Continuity, where we relate the velocities in the (mechanical) system. In the electrical domain, based on the analogy that we are developing we would want to relate the currents in each of the element. In the electrical domain there is a term for this: **Kirchoff's Current Law (KCL)**.

Step 3 in the (mechanical system) process we referred to as Free Body Diagrams, where we relate the forces in the system. For the electrical domain, the analogy would be voltages, and there is also a term for this: **Kirchoff's Voltage Law (KVL).**

In Step 4, we identify the State Variables in the system. For mechanical systems these State Variables were related to the displacements of the springs (x_s) and the velocities of the masses (v_m) in the systems. What would be the analogy in the electrical domain? Answer: charges in the capacitors (q_C) and the currents through the inductors (i_L) would be the natural guesses. But by convention (primarily because it is MUCH easier to measure voltages experimentally than it is to measure charge), **the State Variables in the Electrical Domain are generally assumed to be the Voltage of the Capacitor (V_C) and the currents through the inductors (i_L)!**

Note: This is perhaps the 'largest' difference between the two approaches, and since there is a linear relationship between the voltage and charge of the capacitor from the constitutive relationship, the difference in the grand scheme of things is negligible.

In Step 5, we solve for the State Equations; starting with the derivative of each state variable, we manipulate the expression until the terms on the right-hand side of the equation are suitable. (Recall here what was suitable: system parameters, state variables and derivatives of state variables, and external 'force' and 'velocity' sources.) The exact same procedure is followed for systems in the electrical domain.

Note: In the mechanical domain, we saw that external forces/velocities could include things like the force due to gravity, known velocity of certain masses, etc. What would be the analogous types of 'external' sources in the electrical domain? (Answer: Voltage sources, i.e. batteries, and current sources, for example, constant current sources.)

To illustrate, we will do a simple example in the electrical domain below. Some of the details regarding the application of Kirchoff's Current Law (KCL) and Kirchoff's Voltage Law (KVL) you may recall covering in earlier classes in the curriculum.

EXAMPLE: Consider the electrical system to the right. Find all a) first-order and b) second-order state equations necessary to fully predict the behavior of the system.

Solution: To start, we need to assume a direction for positive flow of the current, which is assumed to go from the negative to positive terminal of the battery (because there is only one loop, the current will continue to flow clockwise in the diagram to the right). By convention, once we identify the direction of the current, we assume that we can label the voltage from each as element as positive-to-negative in the direction of the current (as shown to the right).
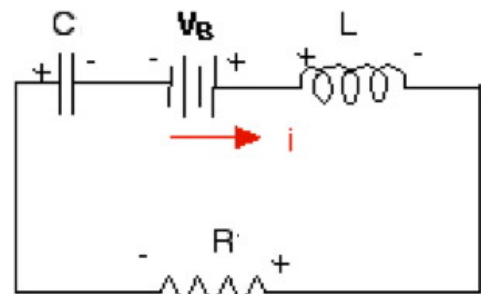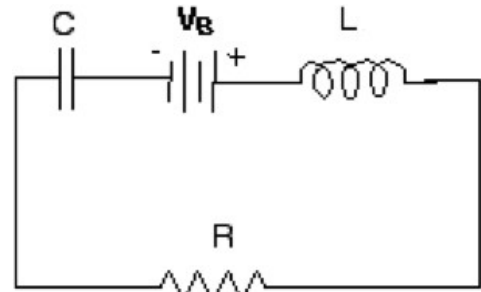
1) **Constitutive Laws**

$$V_C = {}^1\!/_C \, q_C \quad , \quad V_R = R i_R \quad , \quad V_L = L \dot{i}_L$$

2) **Kirchoff's Current Law (analogy to Geometric Continuity)**

$$i_L = i_R = i_C$$

3) **Kirchoff's Voltage Law (analogy to Free Body Diagrams)**

$$V_L + V_R + V_C - V_B = 0$$

4) **Identify State Variables (recall discussion above):**

$$V_C, \quad i_L$$

5) Solve for the (first order) State Equations:

$$V'_C = \frac{1}{C}q'_C = \frac{1}{C}i_C = \frac{1}{C}i_L \qquad \checkmark$$

$$i'_L = \frac{V_I}{L} = \frac{V_B - V_R - V_C}{L} = \frac{V_B - Ri_R - V_C}{L} = \frac{V_B - Ri_L - V_C}{L} \qquad \checkmark$$

You should confirm for yourself that the terms in the final expressions for the State Equations are of a suitable form, and contain only system parameters, state variables or derivatives of state variables, and external 'sources'. There were two first order equations, which was also the case for the spring-mass-damper mechanical system.

To complete the example, we can combine the two first-order equations into a second order equation. We will start with the first of the two first order equations we found above and differentiate both sides:

$$V'_C = \frac{1}{C}\dot{i}_L$$

$$V''_C = \frac{1}{C}i'_L$$

But we can now substitute the second of our first-order equations into the right hand side of the equation above to replace i'_L. Doing so:

$$V''_C = \frac{1}{C}\left(\frac{V_B - Ri_L - V_C}{L}\right)$$

Continuing to simplify, and using the relationship that

$$i_L = i_C = q'_C = C\,V'_C$$

we can show that we ultimately end up with

$$V_C'' = \frac{1}{C}\left[V_B - \frac{V_C}{L} - \frac{RCV_C'}{L}\right]$$

We can re-write this in more standard form as

$$V_C'' + \frac{R}{L}V_C' + \frac{1}{LC}V_C = \frac{1}{LC}V_B$$

This is our second-order differential equation. Note that it is "non-homogeneous" since the right-hand side is non-zero. Compare this to the solution obtained in Example 4 for a spring-mass-damper system given below, and the analogy is clear:

$$x_s'' + \frac{b}{m}x_s' + \frac{k}{m}x_s = g$$

Now that we have found our second-order differential equation, we can determine the behavior of the system using the techniques covered in a class on Differential Equations and highlighted in these ME345 notes.