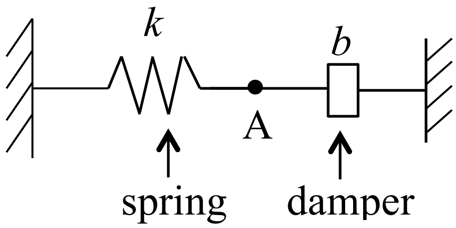
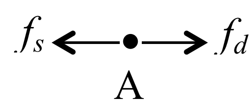
EXAMPLE 1: Spring-damper system



1) Constitutive Laws: $f_x = kx_s$ $\qquad f_d = bv_d$

2) Geometric Continuity: $v_s + v_d$

3) Free Body Diagram (relate forces in system):



$\sum F_x = ma$ (assume positive to right)

$-f_s + f_d = 0$ (because massless connection)

4) Identify State Variables: x_s (only one spring for this example)

5) **Solve for the State Equations**

$x_s{}' = ?$

↑

derivative of state variable

$$x_s{}' = v_s = -v_d = -\frac{f_d}{b} = -\frac{f_s}{b} = -\frac{kx_s}{b}$$

$$x_s{}' = \left(-\frac{k}{b}\right) x_s$$ ← correct form with acceptable terms on right

Thus we have determined the state equation describing the behavior of the system over time. If the system was disturbed, the state equation(s) provides the state of the system as a function of time.

State equation for Example 1: $x_s{}' = \left(-\frac{k}{b}\right) x_s$

This equation can be solved analytically or numerically. To find the analytical solution, use the Method of Undetermined Coefficients (i.e. the 'Guess Method'). To start:

assume $x_s(t) = Ae^{rt}$ is a solution, then…

$$x_s{}'(t) = rAe^{rt}$$

If this "guess" is correct, then it must satisfy the state equation:

$$x_s{}' = -\frac{k}{b} x_s$$

$$r\cancel{A}\cancel{e^{rt}} = -\frac{k}{b}(\cancel{A}\cancel{e^{rt}})$$

$$\boxed{r = -{}^{k}/_{b}}$$ ← this must be true for our guess to work

Thus our solution to the state equation is

$$\boxed{x_s(t) = Ae^{-\frac{k}{b}t}}$$ ← general solution

Here 'A' is a constant that can only be determined from the initial conditions (i.e. the perturbation). There is one unknown; thus we need one initial condition.

In math terminology… we have solved for the general solution of DEQ. If we have the initial conditions, we can solve for the particular solution.

Assume that the initial elongation of the spring is x_s $(t=0) = 2$. What is the particular solution?

$$x_s(t) = Ae^{-\frac{k}{b}t}$$

if at $t=0$, $x_s=2$, then

$$2 = Ae^{-\frac{k}{b}(0)} = A(1)$$

$$A = 2$$

$$\boxed{x_s(t) = 2e^{-\frac{k}{b}t}}$$ ← Particular solution

The Differential Equation can also be solved numerically using techniques covered in MA221. Perhaps the easiest approach is the Euler Method.

The Euler Method can be referred to as a " predictor method". If we know the state of a system at time 't', and we know the rate of how the system is changing at 't', we can predict the state of the system at a small increment in time Δt in the future.

Or, more mathematically, if $[\underline{x}]_n$ is the current state, and $[\underline{x}']_n$ is the rate of change of the system at the current time, then the state of the system at a future time $[\underline{x}]_{n+1}$ is …

$$[\underline{x}]_{n+1} = [\underline{x}]_n + [\underline{x}']_{n+1}(\Delta t)$$

Do NOT let the symbols and the notation confuse you! State in plain English what this equation is telling you!

As a simple example, consider the case of a professor sprinting across the lecture room (at world-class speed). If you know where I am at the current time, and know my velocity (speed and direction), can you predict where I will be in is? If you can, congratulations… you just used the Euler Method!

But be careful … this method is an approximation. If my speed is a constant over the interval Δt, then the solution is exact. But if my velocity is changing over the interval Δt, and you are using my velocity at the beginning of the interval, then this is clearly an approximation! Want a better approximation? Use a smaller time step!

Now let's return to our original state equation,

$$x_s' = \left(-\frac{k}{b}\right)(x_s)$$

Note that here we will assume that we don't know $x_s(t) = 2e^{-\frac{k}{b}t}$. If we already know the analytical solution, why also solve it numerically? (ANSWER: We would not need to!)

For simplicity, let's assume $k=10$ and $b=2$, at that at $t=0$, $x_s=2$. Thus:

$$x_s' = -5x_s$$

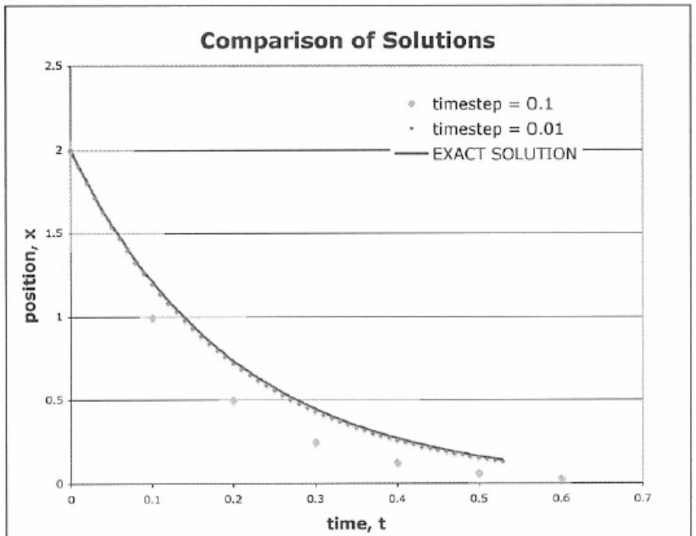
To implement the Euler Method, we can develop and complete the chart below (see appropriate excel file on ME345 Course website).

Plotted are the numerical approximation for two different time steps (Δt=0.1s, Δt=0.01s) and the exact solution. The smaller time step almost perfectly matches the exact solution.

ME 345
Professor Frank Fisher
Euler Method - (approx) numerical solution to DEQs

xs' =	-5	xs
timestep	0.1	

step	time	xs (current)	xs' (current)	xs (future)
1	0	2	-10	1
2	0.1	1	-5	0.5
3	0.2	0.5	-2.5	0.25
4	0.3	0.25	-1.25	0.125
5	0.4	0.125	-0.625	0.0625
6	0.5	0.0625	-0.3125	0.03125
7	0.6	0.03125	-0.15625	0.015625



[same problem, but with a smaller time step]

xs' =	-5	xs
timestep	0.01	

step	time	xs (current)	xs' (current)	xs (future)
1	0	2	-10	1.9
2	0.01	1.9	-9.5	1.805
3	0.02	1.805	-9.025	1.71475
4	0.03	1.71475	-8.57375	1.6290125
5	0.04	1.6290125	-8.1450625	1.54756188
6	0.05	1.54756188	-7.73780938	1.47018378
7	0.06	1.47018378	-7.35091891	1.39667459
8	0.07	1.39667459	-6.98337296	1.32684086
9	0.08	1.32684086	-6.63420431	1.26049882
10	0.09	1.26049882	-6.3024941	1.19747388
11	0.1	1.19747388	-5.98736939	1.13760018
12	0.11	1.13760018	-5.68800092	1.08072018
13	0.12	1.08072018	-5.40360088	1.02668417
14	0.13	1.02668417	-5.13342083	0.97534996
15	0.14	0.97534996	-4.87674979	0.92658246
16	0.15	0.92658246	-4.6329123	0.88025334
17	0.16	0.88025334	-4.40126669	0.83624067
18	0.17	0.83624067	-4.18120335	0.79442864
19	0.18	0.79442864	-3.97214318	0.75470721
20	0.19	0.75470721	-3.77353603	0.71697184
21	0.2	0.71697184	-3.58485922	0.68112325
22	0.21	0.68112325	-3.40561626	0.64706709
23	0.22	0.64706709	-3.23533545	0.61471374
24	0.23	0.61471374	-3.07356868	0.58397805
25	0.24	0.58397805	-2.91989024	0.55477915
26	0.25	0.55477915	-2.77389573	0.52704019
27	0.26	0.52704019	-2.63520094	0.50068818
28	0.27	0.50068818	-2.5034409	0.47565377
29	0.28	0.47565377	-2.37826885	0.45187108
30	0.29	0.45187108	-2.25935541	0.42927753
31	0.3	0.42927753	-2.14638764	0.40781365
32	0.31	0.40781365	-2.03906826	0.38742297
33	0.32	0.38742297	-1.93711484	0.36805182
34	0.33	0.36805182	-1.8402591	0.34964923
35	0.34	0.34964923	-1.74824615	0.33216677
36	0.35	0.33216677	-1.66083384	0.31555843
37	0.36	0.31555843	-1.57779215	0.29978051
38	0.37	0.29978051	-1.49890254	0.28479148
39	0.38	0.28479148	-1.42395741	0.27055191
40	0.39	0.27055191	-1.35275954	0.25702431
41	0.4	0.25702431	-1.28512157	0.2441731
42	0.41	0.2441731	-1.22086549	0.23196444
43	0.42	0.23196444	-1.15982221	0.22036622
44	0.43	0.22036622	-1.1018311	0.20934791
45	0.44	0.20934791	-1.04673955	0.19888051
46	0.45	0.19888051	-0.99440257	0.18893649
47	0.46	0.18893649	-0.94468244	0.17948966
48	0.47	0.17948966	-0.89744832	0.17051518
49	0.48	0.17051518	-0.8525759	0.16198942
50	0.49	0.16198942	-0.80994711	0.15388995
51	0.5	0.15388995	-0.76944975	0.14619545
52	0.51	0.14619545	-0.73097727	0.13888568
53	0.52	0.13888568	-0.6944284	0.1319414
54	0.53	0.1319414	-0.65970698	0.12534433

[here I hard code in the exact solution]

time	position
0	2
0.01	1.90245885
0.02	1.80967484
0.03	1.72141595
0.04	1.63746151
0.05	1.55760157
0.06	1.48163644
0.07	1.40937618
0.08	1.34064009
0.09	1.2752563
0.1	1.21306132
0.11	1.15389962
0.12	1.09762327
0.13	1.04409155
0.14	0.99317061
0.15	0.94473311
0.16	0.89865793
0.17	0.85482986
0.18	0.81313932
0.19	0.77348205
0.2	0.73575888
0.21	0.6998755
0.22	0.66574217
0.23	0.63327354
0.24	0.60238842
0.25	0.57300959
0.26	0.54506359
0.27	0.51848052
0.28	0.49319393
0.29	0.46914058
0.3	0.44626032
0.31	0.42449595
0.32	0.40379304
0.33	0.38409982
0.34	0.36536705
0.35	0.34754789
0.36	0.33059778
0.37	0.31447433
0.38	0.29913724
0.39	0.28454814
0.4	0.27067057
0.41	0.25746981
0.42	0.24491286
0.43	0.23296832
0.44	0.22160632
0.45	0.21079845
0.46	0.20051769
0.47	0.19073832
0.48	0.18143591
0.49	0.17258717
0.5	0.16417
0.51	0.15616333
0.52	0.14854716
0.53	0.14130243