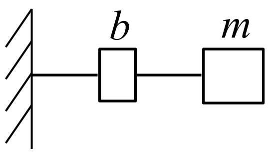
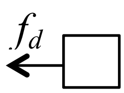
EXAMPLE 2: Mass-Damper System



1) CL $\quad f_d = bv_d$

2) GC $\quad v_d = v_m$

3) 

$$\overrightarrow{\sum F_x}^{+} = ma_m$$

$$-f_d = ma_m$$

4) SV: v_m

5) $v_m' = a_m = -\frac{f_d}{m} = -\frac{bv_d}{m} = -\frac{bv_m}{m}$

$$\boxed{v_m' = -\frac{b}{m} v_m}$$

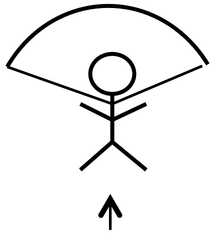
For the analytical solution, again use the Method of Undetermined Coefficients. Guess:

$$v_m(t) = Ae^{rt}$$

Following the procedure from before, you will find

$$v_m(t) = Ae^{-\frac{b}{m}t} \qquad \leftarrow \text{general solution}$$

EXAMPLE 3: Parachute problem



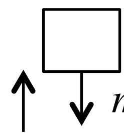
$f_d = bv_d$

(air resistance)

1) CL $\quad f_d = bv_d$

2) GC $\quad v_d = v_m$

3)



$$\downarrow \sum F_y{}^{+} = ma_m$$

$$mg - f_d = ma_m$$

4) SV: v_m

5) State equation(s):

$$v_m' = a_m = \frac{1}{m}\left(mg - f_d\right) = \frac{1}{m}(mg - bv_d)$$

$$= \frac{1}{m}(mg - bv_m)$$

Note: you MUST complete the manipulation of the State Equation(s) until they are in the proper form (with appropriate and acceptable terms on the right hand side). This facilitates the development of analytical and numerical solutions to the problem.

Analytical Solution for Example 3 (Parachute Problem)

Recall the State Equation for problem 3 was

$$v_m' = \frac{1}{m}(mg - bv_m), \text{ or } \ldots$$

$$\boxed{v_m' = g - \frac{b}{m}v_m}$$

Method of Undetermined Coefficients: We add a constant term to our 'guess' because there is a constant term (i.e. gravitational constant) in our Differential Equation. Thus we 'guess':

$$v_m = Ae^{rt} + B \qquad \text{and taking the derivative gives}$$

$$v_m' = rAe^{rt}$$

If our 'Guess' is indeed a solution, then it must satisfy the DEQ such that by we can substitute our guess (and it's derivative) into the DEQ above such that:

$$rAe^{rt} = g - \left(\frac{b}{m}\right)[Ae^{rt} + B]$$

To simplify we can rewrite as:

$$rAe^{rt} = g - \frac{b}{m}Ae^{rt} - \frac{b}{m}B$$

OK, let's step back and think about this. If we find that there are values of *A*, *B*, and *r* such that this equation is true for all values of time *t*, then our 'Guess' works and solves the Differential Equation.

One way to do this is the following thought experiment: The only way that this equation can be true is if the constant and time-dependent terms on the left hand side of the equation are **identical** to those on the right hand side of the equation, which we can express as below:

Constants terms: $0 = g - \frac{b}{m}B$

Time-dependent terms: $rAe^{rt} = -\frac{b}{m}Ae^{rt}$

Thus we have…

$$0 = g - \frac{b}{m}B \quad \Rightarrow \quad \boxed{B = \frac{mg}{b}}$$

$$rAe^{rt} = -\frac{b}{m}Ae^{rt} \quad \Rightarrow \quad \boxed{r = -{}^{b}/_{m}}$$

We can now plug these back into our "Guess" solution (which we know is a solution)…

$$v_m = Ae^{-{}^{b}/_{m}t} + \frac{mg}{b} \qquad \text{Analytical (General) Solution}$$

In mathematical lingo this is again referred to as the General Solution. Why did we not need to solve for A in the equation above? (Answer: Because this solution works for ANY value of A. To solve for A we need to know some initial (or boundary) conditions for our problem, which will yield what is referred to as the Particular Solution.

OK, for a particular solution, we need an initial condition. What is one that makes sense for this problem? Let's use the fact that at t=0, when the parachutist initially jumps out of the plane, what is a good approximation for his initial vertical velocity?

Here it make sense that at $t=0$, $v_m=0$. Thus…

$$0 = A\cancel{e^{-b/m(0)}} + \frac{mg}{b} \quad \Rightarrow \quad \boxed{A = -\frac{mg}{b}}$$

Thus $$\underline{\underline{v_m(t) = -\frac{mg}{b} e^{-b/m t} + \frac{mg}{b}}}$$ **Particular Solution**

It may help to visualize this by re-writing the expression in a different format…

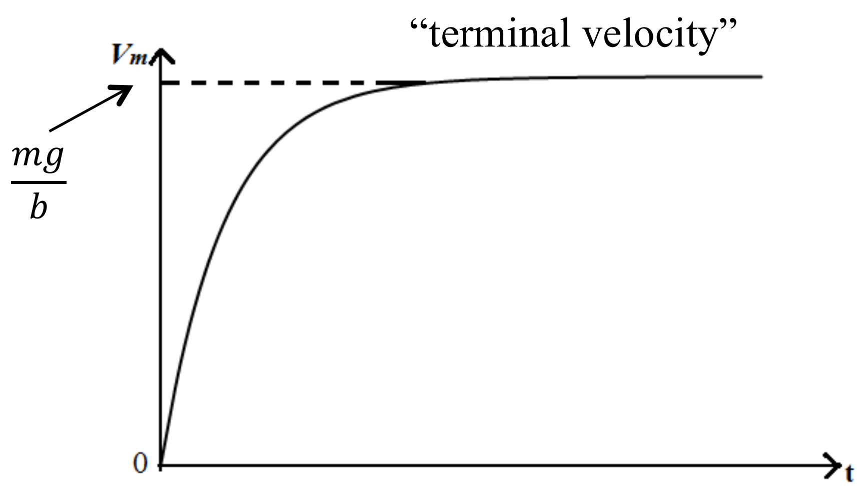
$$\underline{\underline{v_m(t) = \frac{mg}{b}\left(1 - e^{-b/m t}\right)}}$$

We now have a solution, but how do we know that it's sensible? (We may not be able to tell that it's correct, but often times very simple/silly mistakes can be caught by trying to make sense of your solution.)

One easy way to do this is by **sketching a plot of your answer**. Do NOT underestimate the value or power of this sort of analysis. It is meant to be quick and dirty, but can often provide you with some insight into the solution and we well worth the minimal time investment.

Some hints when making such a sketch:

1. be sure to identify any points where the solution crosses the axes
2. identify any 'asymptotic' points (for very large values of a variable)
3. can you identify behavior that is linear, quadratic, exponential, etc?
4. don't waste time making it 'prettier' than it needs to be!



<u>Does this plot make sense when considering the original problem?</u>

1. At $t=0$, $v_m=0$ (this was our initial condition, so needs to be true)

2. As time increases, the velocity increases towards terminal velocity in a "one minus exponential" form

3. At terminal velocity, the drag force (damper) is equal to the gravitational force, so there is <u>no</u> acceleration. If no acceleration, velocity becomes constant.

Does this GUARANTEE that our solution is correct? NO. But if our simple checks don't work, we know that we have an error in our approach.[1]

[1] Mathematicians would refer to this as being "necessary but not sufficient"