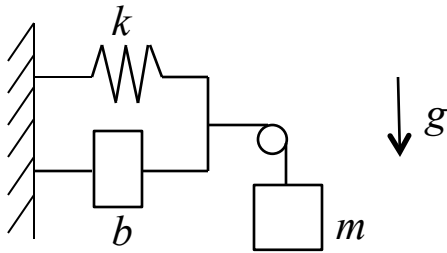


### EXAMPLE 4: Spring-Mass-Damper System with Pulley



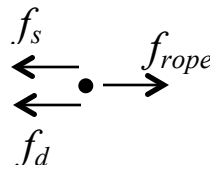
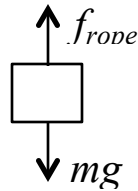
1) CL  $f_x = kx_s \quad f_d = bv_d$

2) GC  $v_s = v_d$

$$v_s = v_m$$

Note that for the GC step here, we assume that positive is ‘to the right’, which corresponds to the mass moving downward. We need to be consistent here; we cannot definite these two directions independently (because they are not independent; they are connected by the pulley)

3) FBD



$$\downarrow \sum F = ma$$

$$-F_{rope} + mg = ma_m$$

$$\overrightarrow{\sum f}^+ = \cancel{m}a = 0 \quad (\text{massless connection})$$

$$F_{rope} - f_s - f_d = 0$$

There are a couple of different ways that one could arrive at the proper equations here. The one that is the slowest is to a FBD at the connection point where the spring and damper connect to the rope going around the pulley (which is then connected to the mass). When writing Newton’s law we assume that the mass of the connection point is negligible, such that  $ma = 0$ . Note that if the mass of this connection point was NOT negligible, it would have been identified with a mass.

4) SV:  $x_s$  ,  $v_m$  (here we have two state variables ... need two state equations)

5) Solve for each state equation separately:

$$\underline{x'_s = v_s = v_m} \quad \checkmark$$

$$v'_m = a_m = \frac{1}{m}(-F_{rope} + mg) = \frac{1}{m}[-(f_s + f_d) + mg]$$

$$= \frac{1}{m}[-kx_s - bv_m + mg]$$

$$\underline{v'_m = \frac{1}{m}[-kx_s - bv_m + mg]} \quad \checkmark$$

Thus we can write the two **FIRST ORDER** state equations as:

$$\begin{cases} x'_s = v_m \\ v'_m = \frac{1}{m}[-kx_s - bv_m + mg] \end{cases}$$

In some cases, for example if we were interested in solving this problem numerically, it would help the process to be able to write the state equations in matrix form. (By now it should be second nature to go from equations to matrix form or vice versa.) So you should be able to prove that we can write these two **first order equations in MATRIX FORM** as

$$\begin{Bmatrix} x'_s \\ v'_m \end{Bmatrix} = \begin{bmatrix} 0 & 1 \\ -k/m & -b/m \end{bmatrix} \begin{Bmatrix} x_s \\ v_m \end{Bmatrix} + \begin{Bmatrix} 0 \\ g \end{Bmatrix}$$

**SECOND ORDER FORM:** On the other hand, to solve this equation analytically it would be easier to **combine these two first order equations into one second order equation of only one of the state variables**. (We can do this because we

know that the behavior of these systems is coupled.) In doing so, we want to make sure that our second-order equation has only ONE state variable (we will likely need to substitute using the equations we found in Steps 1-3).

For example, we can take the first of our two 1<sup>st</sup> order equations and differentiate both sides to get the following:

$$x'_s = v_m$$

$$x''_s = v'_m$$

But we can now substitute the second of our first-order equations into the right hand side of the equation above to replace  $v'_m$ . Doing so:

$$x''_s = \frac{1}{m} [mg - kx_s - bv_m]$$

But since we are writing this equation in terms of  $x_s$ , we want to replace  $v_m$  with a term related to  $x_s$ . From the Geometric Continuity expressions, we found that  $v_m = v_s = x'_s$ , such that

$$x''_s = v'_m = \frac{1}{m} [mg - kx_s - bx'_s]$$

We can re-write this in more standard form as

$$x''_s + \frac{b}{m} x'_s + \frac{k}{m} x_s = g$$

This is our second-order differential equation. Note that it is “non-homogeneous” since the right-hand side is non-zero.