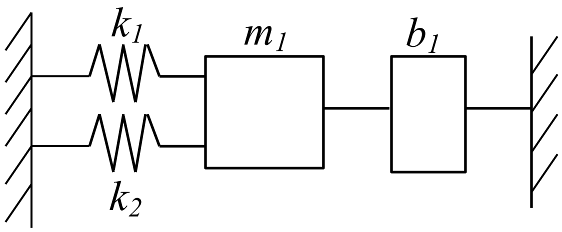
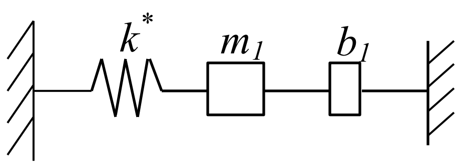
EXAMPLE 6: Spring-Mass-damper system



Find: state equations

Note: On inspection, you could see that k_1 and k_2 are in parallel, and equivalent to the system below where $k^* = k_1 + k_2$. In the notes below we will instead solve the equivalent system and substitute in the definition of k^* at the end. Alternatively, one could not make the substitution and instead solve for the state equations using the same procedure as earlier. The only different here is that there are only two state variables in the system (and not three as one might have initially identified, since x_{s1} and x_{s2} are identical.

EQUIVALENT SYSTEM



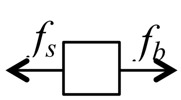
$$\boxed{k^* = k_1 + k_2}$$

Solving for the State Equations

1. $f_s = k^* x_s$ $\qquad$ $f_b = b_1 v_d$

2. $v_s = v_m$ $\qquad$ $v_b = -v_m$

   Note that as the damper expands the mass must be moving to the left, hence the negative sign

3. 

$$\sum F_x = m\, a_x$$

$$-f_s + f_b = m a_m$$

4. SVs: x_s (note $x_s = x_{s1} = x_{s2}$)

v_m

5. $x_s{}' = v_s = v_m$

$$v_m' = a_m = \frac{1}{m}\left(-f_s + f_b\right) = \frac{1}{m}(-k^* x_s + b v_d) = \frac{1}{m}(-k^* x_s - b v_m)$$

As we did previously, we can combine the two first order state equations into a single order state equation in terms of x_s:

$$x_s{}'' = v_m' = \frac{1}{m}(-k^* x_s - b v_m)$$

$$x_s{}'' = v_m' = \frac{1}{m}(-k^* x_s - b x_s{}')$$

need in terms of x_s here

2nd order state equation → $\boxed{m x_s{}'' + b x_s{}' + (k_1 + k_2) x_s = 0}$

Note here we substituted in for $k^* = k_1 + k_2$

Now that we have our second order differential equation, the goal is to solve it; here we will solve it analytically. There are likely a few ways that you have learned to do this; one way is to use the Laplace transform approach. You are of course free to use your own favorite way to solve this equation; my personal favorite is the Method of Underdetermined Coefficients (which I like to call the "Guess" Method) that we have used previously in this class:

To start, we guess that $x_s(t) = Ae^{rt}$

$$x_s' = rAe^{rt}$$

$$x_s'' = r^2Ae^{rt}$$

If this is the solution, then these must satisfy DEQ…

$$m(r^2\cancel{Ae^{rt}}) + b(r\cancel{Ae^{rt}}) + (k_1 + k_2)\cancel{Ae^{rt}} = 0$$

$$mr^2 + br + (k_1 + k_2) = 0$$

This important equation is referred to as the characteristic equation. To solve, need to find the values of ‘r’ that satisfy the characteristic equation. To find these, use quadratic formula.

$$r_{1,2} = \frac{-b \pm \sqrt{b^2 - (4)(m)(k_1 + k_2)}}{2m}$$

Which we can write in a slightly different form as

$$r_{1,2} = -\frac{b}{2m} \pm \sqrt{\left(\frac{b}{2m}\right)^2 - \frac{(k_1 + k_2)}{m}}$$

Thus the value of r_1, r_2 are dependent on the system parameters. Depending on these values, we’ll get different system responses (see general vibrations book for more detail).

It’s beyond the scope of this class, but the “damping ratio” ζ is derived as

$$\zeta = \frac{b}{2\sqrt{km}}$$

Depending on the values of the system parameters m, b, and k, there are there are four categories (or 'types' of responses) that we can get from the system, depending on the value of the damping ratio ζ:

Case 1— undamped $b=0, \zeta=0$

Case 2 — under-damped $0<\zeta<1$

Case 3 — critically-damped $\zeta=1$

Case 4 — over-damped $\zeta>1$

An example of the different types of response for these different cases is shown below.

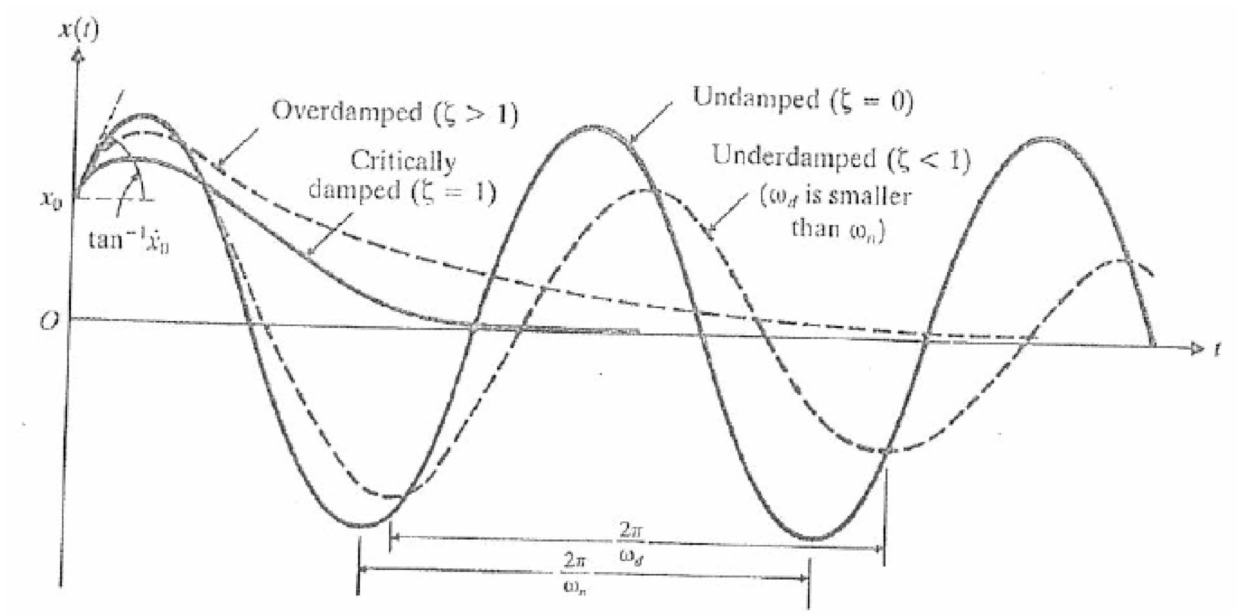


Figure 2.16. Comparison of motions with different types of damping.

"Mechanical Vibrations", 2nd Edition, S.S.Rao. © 1990

A few notes about the behavior shown in the figure:

1. If the system is undamped, there is no energy loss in the system and theoretically the system would continue to vibrate. Real systems always have losses in the system

2. If the system is underdamped, the system will vibrate but with increasingly smaller amplitude over time. Note that there is a (generally slight) decrease in the underdamped case (in comparison to the un-damped system; this is referred to as the damped frequency). The relationship between the damped frequency ω_d and the undamped natural frequency ω_0 can be written as

$$\omega_d = \omega_0\sqrt{1 - \zeta^2}$$

   In a classical vibrations course this would be covered in more detail.

   Note: a good check would be to consider with the equation for damped frequency above is sensible. If we consider the four cases that were discussed above:

   - Case 1: Undamped: $\zeta=0$, and $\omega_d = \omega_0$. This makes sense
   - Case 2: Underdamped: $0<\zeta<1$. ω_d will be smaller than ω_0
   - Case 3: Critically damped: $\zeta=1$. Suggests no oscillation (and hence no frequency). Makes sense.
   - Case 4: Overdamped: $\zeta>1$. This would result in the square root of a negative number; expression is no valid in this case

3. Notice the difference between critically damped and overdamped. In both cases the system goes to zero with no oscillation, but the systems 'gets' to zero fast in the critically damped case. This is quite useful in a number of applications, for example vibration absorbers, where we want to 'damp out' the vibration as quickly as possible. (One common example is the shock absorbers in your car.) For a given value of *k* and *m*, there is a 'perfect' damping value that will result in the system being critically damped and resulting in optimal performance.

4. For the overdamped case, there is too much damping in the system, and it cannot get to zero as quickly as the critically damped system can. I personally think of this system as "sluggish".