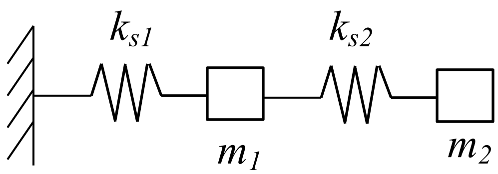
EXAMPLE 7: EIGENVALUE PROBLEM EXAMPLE

x→

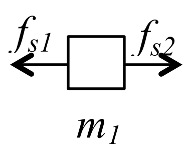


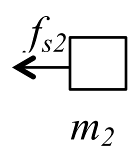
1) CL: $f_{s1} = k_1 x_{s1}$ $\qquad$ $f_{s2} = k_2 x_{s2}$

2) GC: $v_{s1} = v_{m1}$

$v_{s2} = v_{m2} - v_{m1}$

3) FBD:





$$\overrightarrow{\sum F_x}^{+} = m_1 a_{m1} \qquad \overrightarrow{\sum F_x}^{+} = m_2 a_{m2}$$

$$-f_{s1} + f_{s2} = m_1 a_{m1} \qquad -f_{s2} = m_2 a_{m2}$$

4) State variables: $x_{s1}, x_{s2}, v_{m1}, v_{m2}$ (Four SV's here!)

5) Solve for the state equations for each variable

$$x'_{s1} = v_{s1} = v_{m1} \quad \surd$$

$$x'_{s2} = v_{s2} = v_{m2} - v_{m1} \quad \surd$$

$$v'_{m1} = a_{m1} = \frac{1}{m1}\left(-f_{s1} + f_{s2}\right) = \frac{1}{m1}\left(-k_1 x_{s1} + k_2 x_{s2}\right) \quad \surd$$

$$v'_{m2} = a_{m2} = \frac{1}{m2}\left(-f_{s2}\right) = \frac{1}{m2}\left(-k_2 x_{s2}\right) \quad \surd$$

Wow, that was one of our easier examples of solving for the first order state equations! Who said that this 'Eigenvalue Problem' was complicated? Although this is the first problem where we have four first-order state equations, so maybe that will be part of the challenge. Moving on…

Recall that in a past example we wrote the **first order equations in matrix form**. We can repeat that process here…

$$\begin{Bmatrix} x'_{s1} \\ x'_{s2} \\ v'_{m1} \\ v'_{m2} \end{Bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ -{}^{k_1}/_{m_1} & {}^{k_2}/_{m_2} & 0 & 0 \\ 0 & -{}^{k_2}/_{m_2} & 0 & 0 \end{bmatrix} \begin{Bmatrix} x_{s1} \\ x_{s2} \\ v_{m1} \\ v_{m2} \end{Bmatrix}$$

However, it will be <u>more useful</u> to **write these equations in second-order form**. We will do this in terms of the velocities of the masses. (Recall that as in the previous example where we did this, we may need to re-write some of the previous state variables in terms of acceptable state variables for the second order system; here the velocities of the masses.)

$$v''_{m1} = \frac{d}{dt}(v'_{m1}) = \frac{1}{m1}(-k_1 x'_{s1} + k_2 x'_{s2})$$

$$= \frac{1}{m1}[-k_1 v_{s1} + k_2 v_{s2}]$$

$$= \frac{1}{m1}[-k_1 v_{m1} + k_2(v_{m2} - v_{m1})] \quad \surd$$

$$v''_{m2} = \frac{d}{dt}(v'_{m2}) = \frac{1}{m2}(-k_2 x'_{s2})$$

$$= \frac{1}{m2}[-k_2 v_{s2}]$$

$$= \frac{1}{m2}[-k_2(v_{m2} - v_{m1})] \quad \surd$$

We can now write these **second order equations in matrix form** as

$$\begin{Bmatrix} v''_{m1} \\ v''_{m2} \end{Bmatrix} = \begin{bmatrix} -\dfrac{k_1}{m_1} - \dfrac{k_2}{m_2} & {k_2}/{m_1} \\ {k_2}/{m_2} & -{k_2}/{m_2} \end{bmatrix} \begin{Bmatrix} v_{m1} \\ v_{m2} \end{Bmatrix}$$

The next step is to solve our problem analytically. Recall when we did this before for the case of a single second-order state equation, we found the *characteristic equation*. But now we have to solve two coupled second order-differential equations simultaneously! This is in the form of the classical **eigenvalue problem.** In the past examples, we've used the Method of Undetermined Coefficients (i.e the "Guess Method") to determine the analytical solution. We can proceed with this procedure here, with some minor modifications to account for the fact that we are now trying to guess the solution to multiple equations simultaneously.

To simplify the notation here, we will write our 2x2 matrix system above in the following shorthand form:

$$\underline{v''_m} = [A]\underline{v_m}$$

where $[A] = 2\times2$ matrix, and v''_m, and v_m are 2×1 matrices such that

$$[A] = \begin{bmatrix} -\dfrac{k_1}{m_1} - \dfrac{k_2}{m_2} & {k_2}/{m_1} \\ {k_2}/{m_2} & -{k_2}/{m_2} \end{bmatrix}, \qquad \underline{v''_m} = \begin{Bmatrix} v''_{m1} \\ v''_{m2} \end{Bmatrix} = \begin{Bmatrix} v''_{m1}(t) \\ v''_{m2}(t) \end{Bmatrix},$$

$$\underline{v_m} = \begin{Bmatrix} v_{m1} \\ v_{m2} \end{Bmatrix} = \begin{Bmatrix} v_{m1}(t) \\ v_{m2}(t) \end{Bmatrix}$$

The notation may look complicated but don't let it confuse you. We want to find $\underline{v_m}$(t) – this is simply the velocities of each of the two masses as a function of time.

To continue with the Method of Undetermined Coefficients, we need to "Guess" a solution. But since we need to "Guess" two simultaneous solutions, we will guess:

$$\underline{v_m}(t) = \begin{Bmatrix} v'_{m1}(t) \\ v'_{m2}(t) \end{Bmatrix} = \underline{v_i} e^{\lambda t}$$

Keep in mind that $\underline{v_m}$ is a 2x1 <u>vector</u>, since we need to guess the two solutions simultaneously. (In past examples we've guessed a solution in the form $Ae^{\lambda t}$ where *A* was a constant. We need a more 'complicated' guess here for a more complicated problem.) Repeating the previous process, we can then differentiate our guess to find:

<u>Guess</u> $\underline{v_m}(t) = \underline{v_i} e^{\lambda t}$

$$\underline{v_m}'(t) = \lambda \underline{v_i} e^{\lambda t}$$

$$\underline{v_m}''(t) = \lambda^2 \underline{v_i} e^{\lambda t}$$

If our "guess" solution works, then we need to plug it back into our original equation

Original equation:	$\underline{v''_m} = [A]\underline{v_m}$
Substituting:	$\lambda^2 \underline{v_i} e^{\lambda t} = [A]\underline{v_i} e^{\lambda t}$
Cancelling exponential term	$\lambda^2 \underline{v_i} = [A]\underline{v_i}$
Subtracting over:	$([A] - \lambda^2 [I])\underline{v_i} = 0$

where *I* is the identity matrix (with ones on the diagonals, necessary to introduce so that we are subtracting 2x2 matrices). This last expression is referred to as the **eigenvalue problem.**

The eigenvalue problem is quite relevant in all fields of science and engineering. (In addition to this Mechanical System, eigenvalue problems are also found in: studies of predator-prey biological systems used to study extinction patterns; War Games simulations to estimate troop casualties; and in image-based Facial Recognition systems, to name a few examples.) Because of this the eignenvalue problem is written in many different ways, often depending on the field (or sub0-field); for example, rather than 'λ' people sometimes use 'ω' or 'α'. But although written in different forms, what the eigenvalue problem "means" is the same, which we can express in words as:

"Given a matrix [A], can we find scalar quantities λ_i and vectors $\underline{v_i}$ that satisfy (i.e. solve) this eigenvalue problem"

More terminology: the λ_i are scalars and are referred to as the eigenvalues, while $\underline{v_i}$ are vectors and are referred to as the eigenvectors, or mode shapes. Eigenvalues are related to the natural frequencies of the system. Each eigenvalue has a distinct (different) mode shape.

OK, now back to our analytical solution. Recall that our eigenvalue problem was of the form

$$([A] - \lambda^2[I])\underline{v_i} = 0$$

This equation can only be solved if the following two conditions are *both* met:

1) det $([A] - \lambda^2[I]) = 0$, and <u>then</u>

2) if appropriate $\underline{v_i}$ are selected.

(recall that *det* is the determinant of the system as you have talked about previously in calculus). Going back to our original problem, recall that our *A* matrix (which was determined based on how the springs and masses were connected in the system) was found to be:

$$A = \begin{bmatrix} -\frac{k_1}{m_1} - -\frac{k_2}{m_2} & {k_2}/{m_1} \\ {k_2}/{m_2} & -{k_2}/{m_2} \end{bmatrix}$$

To make our discussion easier, let's assume the following values for our system parameters:

$$k_1 = 3, k_2 = 2$$

$$m_1 = 1, m_2 = 1$$

Substituting these values into our *A* matrix we find that

$$A = \begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix}$$

and thus

$$[A] - \lambda^2[I] = \begin{bmatrix} -5 - \lambda^2 & 2 \\ 2 & -2 - \lambda^2 \end{bmatrix}$$

The first of our two conditions to find a solution is that the determinant of $[A] - \lambda^2[I] = 0$. Assuming you recall how to take the determinant of a 2x2 matrix, for what value of λ is this determinant equal to zero?

$$(-5 - \lambda^2)(-2 - \lambda^2) - 4 = 0$$

$$10 + 5\lambda^2 + 2\lambda^2 + \lambda^4 - 4 = 0$$

$$\lambda^4 + 7\lambda^2 + 6 = 0$$

$$(\lambda^2 + 6)(\lambda^2 + 1) = 0$$

$$\lambda_1 = \pm i \qquad\qquad \lambda_2 = \pm\sqrt{6}i$$

By convention, we call the smallest value the " first" eigenvalue, the second smallest value the second eigenvalue, etc.

These are the only possible eigenvalues that allow the necessary determinant to be equal to zero. So we've found part of our solution. The next step is to solve for the appropriate eigenvectors (mode shapes).

If we use <u>these</u> eigenvalues, what values of $\underline{v_i}$ do we need? We will do this on a case-by-case basis, by substituting the eigenvalues that we just found back into our eigenvalue problem formulation as illustrated below.

<u>Case 1:</u> $\boldsymbol{\lambda_1 = \pm i}$

$$\lambda_1 = \pm i, \lambda^2 = -1$$

$$([A] - \lambda^2[I])\underline{v_1} = 0$$

$$\begin{bmatrix} -5-(-1) & 2 \\ 2 & -2-(-1) \end{bmatrix} \begin{Bmatrix} v_{11} \\ v_{12} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix} \begin{Bmatrix} v_{11} \\ v_{12} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Note: We'll use the notation vij to denote the elements of the eigenvector, where *i* represents the eigenvalue (in this case, we are looking using the value for the first eigenvalue) and *j* represents the row that the component refers to: this here we want to determine v11 and v12

Note that these two equations (as represented by the matrix problem above) are NOT independent! There is an infinite number of combinations of $v_{11}$and v_{12} that will work! For example, all of these combinations work (try them if you don't believe me!):

$$v_{11} = 1, v_{12} = 2$$

$$v_{11} = 2, v_{12} = 4$$

$$v_{11} = -7.6, v_{12} = -15.2$$

$$v_{11} = {}^{1}\!/_{\sqrt{5}}, v_{12} = {}^{2}\!/_{\sqrt{5}}$$

They all work! That is fine. What is MORE IMPORTANT than the values of the eigenvalues is the relative magnitude between the two, which tells us the 'direction' of the vector (as opposed to the magnitude of the vector). In all of the possible solutions above, note that v_{12} is exactly two times that of v_{11} – **that** is what is important. We will see later below why which values you choose doesn't matter; ultimately you will get the same solution.

OK, so if it doesn't matter exactly which correct combination of the components of the eigenvector we choose, how do we find one that works? Here is the easiest (i.e. best) way: set $v_{11} = 1$, then find out which values of v_{12} works. Doing this for the above problem you can show that

$$\underline{v_1} = \begin{Bmatrix} v_{11} \\ v_{12} \end{Bmatrix} = \begin{Bmatrix} 1 \\ 2 \end{Bmatrix}$$

Note: In some fields, such as mathematics, the fact that there can be more than one suitable answer for the eigenvector is disconcerting. One way to overcome this hurdle is to *normalize* the vector such that it's magnitude is equal to one. (In this way everyone will find the same exact eigenvector even if they picked different initial eigenvalues.) To normalize, one can divide by the square of the sum of the components ($\sqrt{1^2 + 2^2}$=$\sqrt{5}$), such that

$$\underline{v_1} = \begin{Bmatrix} v_{11} \\ v_{12} \end{Bmatrix} = \begin{Bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{Bmatrix}$$

So what does this eigenvector *mean*, and how is it related to the eigenvalue that it corresponds to? If you think of the eigenvalue as a frequency, then the eigenvector tells us the *mode shape* that corresponds to that frequency. In this case, it tells us that at this frequency the velocities of the two masses are related by a factor of =2; i.e. that the velocity of the second mass is 2x that of the first mass (note that they will always be in the same direction; if one is going in the 'negative' direction as defined in the problem then the other will be going in the same direction).

To recap, we've just found the eigenvalue that corresponds to the first eigenvector. We need to repeat the process for the remaining eigenvalues as outlined below.

<u>Case 2:</u> $\boldsymbol{\lambda_2 = \pm\sqrt{6}i}$

We will skip some of the details here and leave it to you to go through the process to find the appropriate eigenvector that is associated with the second eigenvalue.

$$\lambda_2 = \pm\sqrt{6}i, \lambda^2 = -6$$

$$([A] - \lambda^2[I])\underline{v_2} = 0$$

$$\begin{bmatrix} -5-(-6) & 2 \\ 2 & -2-(-6) \end{bmatrix} \begin{Bmatrix} v_{21} \\ v_{22} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{Bmatrix} v_{21} \\ v_{22} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\underline{v_2} = \begin{Bmatrix} v_{21} \\ v_{22} \end{Bmatrix} = \begin{Bmatrix} 1 \\ -{}^1\!/_2 \end{Bmatrix}$$

Note that in this case the velocities are in opposite directions; if one mass is moving to the right, the other will be moving to the left (and vice versa).

So to recap: we have found that

$$\textbf{Case 1} \qquad \lambda_1 = \pm i\,, \qquad \underline{v_1} = \begin{Bmatrix} 1 \\ 2 \end{Bmatrix}$$

$$\textbf{Case 2} \qquad \lambda_2 = \pm\sqrt{6}i\,, \quad \underline{v_2} = \begin{Bmatrix} 1 \\ -1/_2 \end{Bmatrix}$$

These are related to the natural frequencies!

"eigenvectors"

"mode shapes"

"normal modes"

OK, so we've found all the eigenvalues and eigenvectors in the problem. A key element of this solution was the *A* matrix, which was comprised of various system parameters (spring constants, masses) and whose arrangement was dependent on how the elements were organized in the real system. (In other words, by changing the values we assumed for the spring constants and the masses we would find different eigenvalues and different eigenvectors.) But how do these eigenvalues and eigenvectors relate to the solution for our problem at hand?

Recall that we started this process by using the Method of Undetermined Coefficients and by assuming that out solution was of the form

$$\underline{v_m}(t) = \underline{v_i} e^{\lambda t} \quad \leftarrow \text{this was our "guess"}$$

Substituting the values that we found, and remembering from our calculus class that adding solutions to a differential equation gives us an expression that is also a solution to the differential equation, we can write:

$$\underline{v_m}(t) = \begin{Bmatrix} 1 \\ 2 \end{Bmatrix} e^{\pm it} + \begin{Bmatrix} 1 \\ -1/_2 \end{Bmatrix} e^{\pm\sqrt{6}it}$$

Long (Cool?) Note: The above form, while correct, is not in itself very useful. However, you may recall from calculus that we can rewrite this expression in terms of *sin* and *cos* by using what is known as **Euler's Formula for Complex Analysis** (yes, that Euler!), which is given as

$$e^{ix} = \cos x + i \sin x$$

If one goes back to the Euler Formula, when x = π the Euler Formula gives us $e^{i\pi} + 1 = 0$, which is known as the Euler Identity. The Euler Identity, which is comprised of five of the most fundamental mathematical constants (0, 1, π, *e*, and *i*), has been referred to by different sources as the "most beautiful theorem in mathematics" and the "greatest equation ever". (Wikipedia)

Thus using the Euler Formula for Complex Analysis, we can rewrite our solution as:

$$\underline{v_m}(t) = \begin{Bmatrix} 1 \\ 2 \end{Bmatrix} (a_1 \cos t + b_1 \sin t) + \begin{Bmatrix} 1 \\ -{}^{1}\!/_{2} \end{Bmatrix} \left(a_2 \cos\sqrt{6}\, t + b_2 \sin \sqrt{6} t\right)$$

where a_1, b_1, a_2, b_2 are constants which depend on initial/boundary conditions of the problem. Does it make sense that there are four such constants? Yes, because initially there were 4 state variables, which led to four 1st order state equations; this is all connected.

In mathematical terms, the equation above is the *general solution*. For a particular problem, we can use known initial/boundary conditions to solve for the *particular solution*.

In the solution above there are two **modes** (which correspond to the 'cases' that we looked at before). The first mode has a frequency of 1 and a mode shape (eigenvector) of [1;2]. The second mode has a frequency of $\sqrt{6}$ and a mode shape (eigenvector) of [1;-0.5].

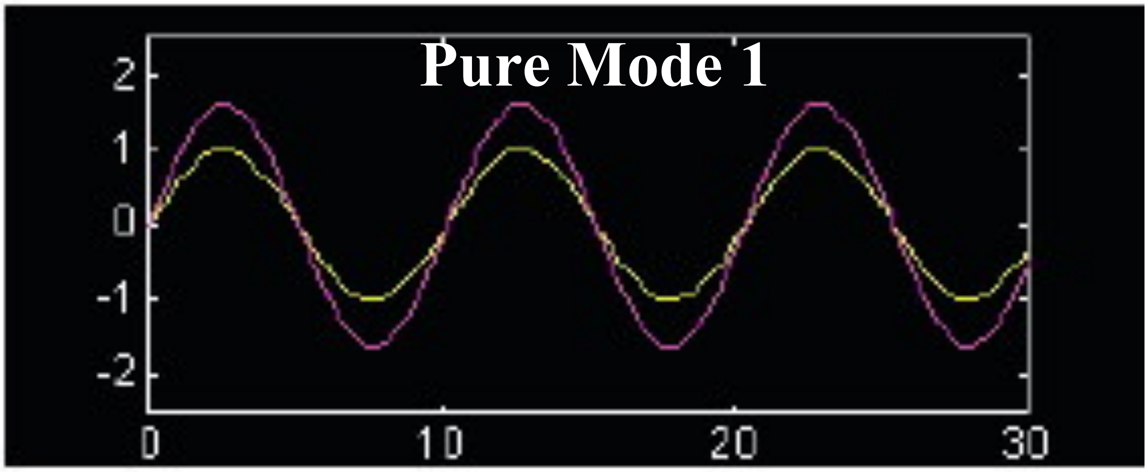
MODE 1 $\begin{Bmatrix} 1 \\ 2 \end{Bmatrix} f(t)$

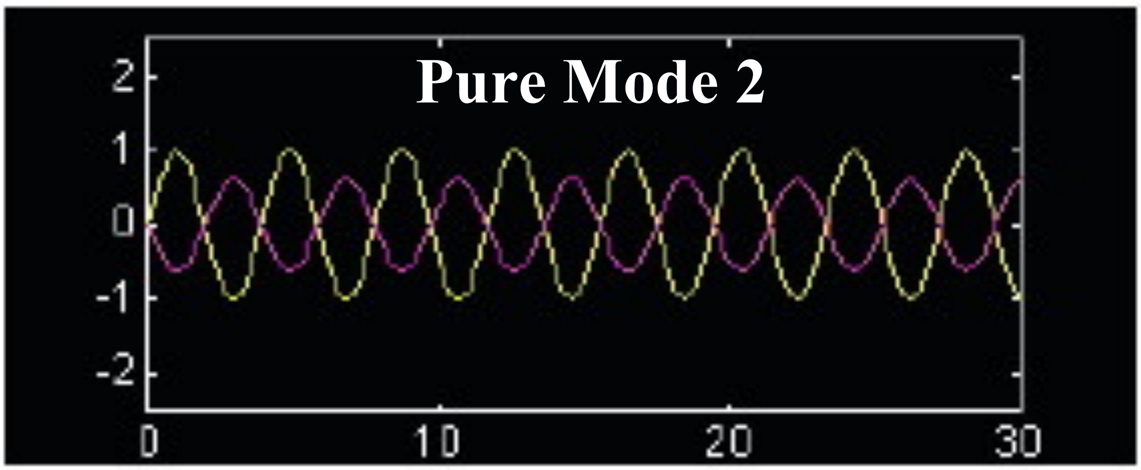
- Velocities are the same sign
- They are moving “in unison” (in the same direction), although mass 2 moving twice as fast as mass 1

MODE 2 $\begin{Bmatrix} 1 \\ -{}^{1}\!/_{2} \end{Bmatrix} f\left(\sqrt{6}t\right)$

- Velocities are the opposite sign
- Masses are moving in opposite directions, with mass 2 speed 50% that of mass 1.

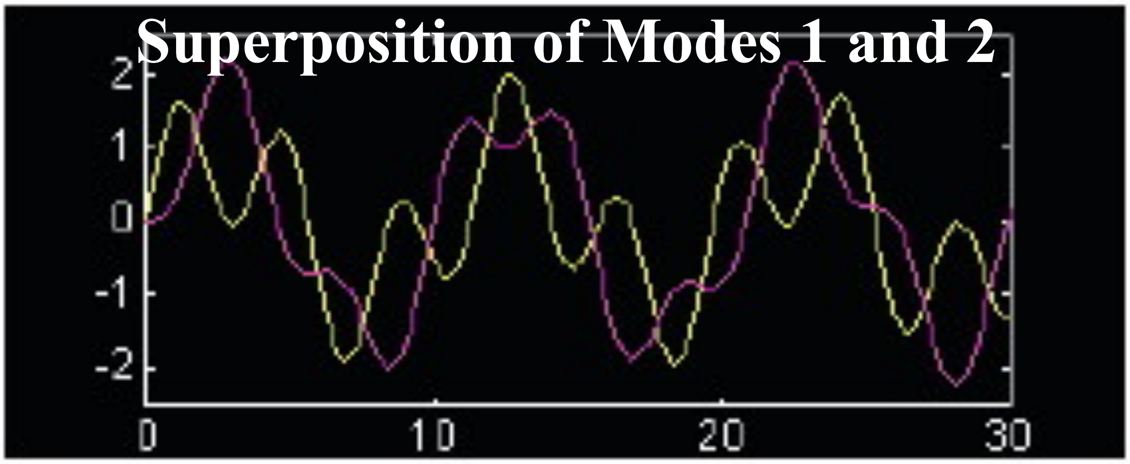
Under very specific conditions (i.e. initial conditions), the system could move in a way such that the behavior was purely Mode 1, which would result in a very smooth sinusoidal behavior at a frequency of 1. Likewise, under a different set of initial conditions we could achieve pure Mode 2 behavior, such that the behavior was again sinusoidal but at a different (and higher) frequency of $\sqrt{6}$ as demonstrated below.





In the figures above be sure to notice the following: 1) frequency of Mode 2 is higher that Mode 1; 2) in Mode 1 behavior, when the velocity of one mass is positive, the other is also positive (and vice versa); 3) in Mode 2 behavior, when one velocity is greater than zero, the other velocity is negative (i.e. moving in the negative direction) and vice versa; and 4) the [1;2] and [1;-0.5] Eigenvectors are demonstrated.

Now, if we do NOT have just the right set of initial conditions, then the behavior of the system will not demonstrate the behavior of just one mode or another mode, but will instead be a *superposition* of the two modes (and just these two modes, only). Even for the simple example above, this sort of behavior can *look* quite chaotic, as shown below:



This is a key finding. Even though the behavior of the system above *appears* to be quite chaotic (or crazy, ugly, messy, etc, depending on your word choice), we know that the velocity of each mass in the system and they can be written in terms of just the eigenvalues and eigenvectors. Moreover, we know that, given the right initial conditions, we can cause the system to operate in a particular mode (with a corresponding a particular frequency).

To complete the discussion, let's give ourselves a set of initial conditions and practice **finding the particular solution**. Here we'll assume we know the initial conditions:

$$v_{m1}(t=0)=0 \qquad v_{m1}'(t=0)=3.45$$

$$v_{m2}(t=0)=0 \qquad v_{m2}'(t=0)=.775$$

Note: What do these initial conditions mean? How could one take the system that we are working with and impose these very specific initial conditions?

Rewriting the general solution we found for the velocities of the masses as

$$\underline{v_m}(t) = \begin{Bmatrix} 1 \\ 2 \end{Bmatrix} (a_1 \cos t + b_1 \sin t) + \begin{Bmatrix} 1 \\ -1/2 \end{Bmatrix} \left(a_2 \cos\sqrt{6}\, t + b_2 \sin \sqrt{6} t\right)$$

we can easily take the derivative with respect to time, which gives us:

$$\underline{v_m}'(t) = \begin{Bmatrix} 1 \\ 2 \end{Bmatrix} (-a_1 \sin t + b_1 \cos t) + \begin{Bmatrix} 1 \\ -1/2 \end{Bmatrix} \sqrt{6}\left(-a_2 \sin \sqrt{6} t + b_2 \cos\sqrt{6}\, t\right)$$

We have four unknowns (a_1, b_1, a_2, b_2), so we need four independent equations, which we can find by using the initial conditions at t=0. At t=0,

$\boldsymbol{v_m(t=0)}$

$$\begin{Bmatrix} 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 2 \end{Bmatrix} (a_1) + \begin{Bmatrix} 1 \\ -1/2 \end{Bmatrix} (a_2)$$

$$\begin{Bmatrix} 0 \\ 0 \end{Bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & -1/2 \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix} \Rightarrow \begin{matrix} a_1 = 0 \\ a_2 = 0 \end{matrix}$$

$\boldsymbol{v_m'(t=0)}$

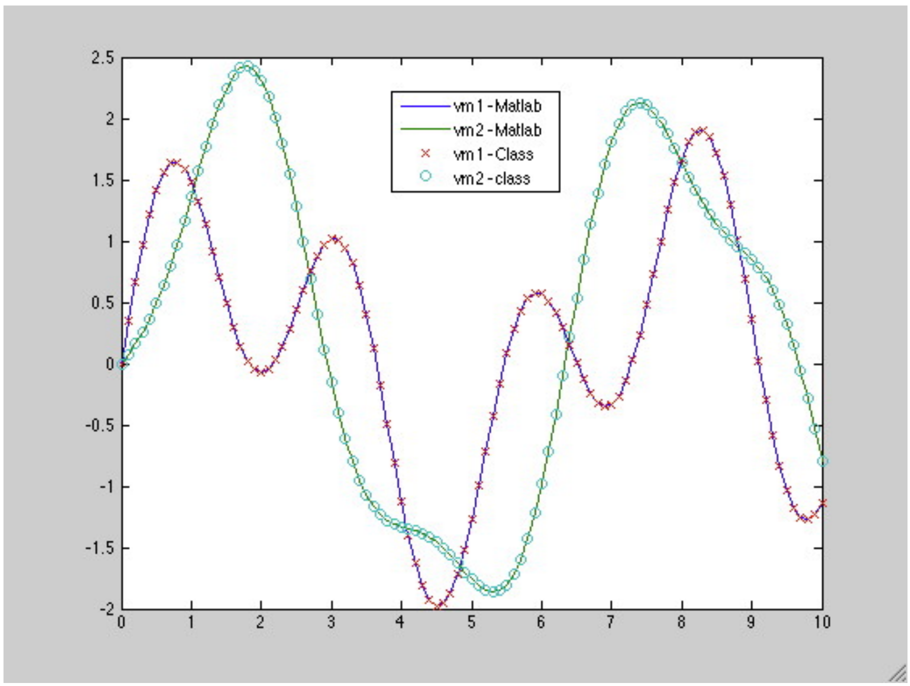
$$\begin{Bmatrix} 3.45 \\ .775 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 2 \end{Bmatrix} (b_1) + \begin{Bmatrix} 1 \\ -1/2 \end{Bmatrix} \sqrt{6}(b_2)$$

$$\begin{Bmatrix} 3.45 \\ .775 \end{Bmatrix} = \begin{bmatrix} 1 & \sqrt{6} \\ 2 & -\sqrt{6}/2 \end{bmatrix} \begin{Bmatrix} b_1 \\ b_2 \end{Bmatrix} \Rightarrow \begin{matrix} b_1 = 1 \\ b_2 = 1 \end{matrix}$$

Plugging these values for (a_1, b_1, a_2, b_2) into the general solution will give us the particular solution, which can be written as:

$$\underline{v_m}(t) = \begin{Bmatrix} v_{m1}(t) \\ v_{m2}(t) \end{Bmatrix} = \begin{Bmatrix} 1 \\ 2 \end{Bmatrix} \sin t + \begin{Bmatrix} 1 \\ -1/2 \end{Bmatrix} \sin\sqrt{6}\, t$$

A plot of the velocities is shown below (it is the same plot that we saw before, which is a superposition of the two modes):



Matlab lends itself quite nicely to solving these types of problems (it has built-in functions to determine the eigenvalues and eigenvectors of the A matrix that we found before). An example Matlab code that can be used for this case is shown below.

```
% Frank Fisher
% ME345
% Eigenvalue problem... solution via Matlab
% This is for the two spring and two mass system

%            --------                ------------
%    k1      |       |       k2      |           |
%--/\/\/\---|   m1   |------/\/\/\---|    m2     |
%            |       |               |           |
%            --------                -------------

clear
clc
close all

% INPUT system parameters (in class, used k1=3, k2=2, m1=1, m2=1)
k1=3;       k2=2;       m1=1;       m2=1;

% Input initial conditions (assumed initial velocity and accel of masses)
vm1_t0=0;           % init velocity of m1
vm2_t0=0;           % init velocity of m2
am1_t0=3.45;        % init accel of m1
am2_t0=.775;        % init accel of m2

% If want to set specific values for the constants (to get normal mode
% behavior) set this

normal_mode=0;  % if set equal to 1, use THESE constants in final solution

%%%%%%%% DO NOT CHANGE ANYTHING BELOW THIS LINE %%%%%%%%

A = [-k1/m1 - k2/m1, k2/m1;
    k2/m2, -k2/m2];             %this is the A matrix

[e_vectors, e_values]=eig(A);   %Matlab built-in command
                                %e_vectors - columns give the eigenvectors
                                %e_values - diagonals given the eigenvalues

evect1=e_vectors(:,1)          %this is one of the eigenvectors
evect2=e_vectors(:,2)          %this is the other eigenvector
eval1=-e_values(1,1)           %one of the eigenvalues (note: negative here)
eval2=-e_values(2,2)           %one of the eigenvalues (note: negative here)

%Following the notation of class, once we have these values we can write
%the velocities of the masses as a function of time (and unknown constants
%based on the initial conditions).

%To find the accelerations, also need to take the derivatives of these
%velocities (See class notes)

%To solve for unknowns a1, a2, b1, b2, use boundary conditions. Will solve
%in matrix form. Note: assuming that all BCs are at t = 0!
temp_matrix1=[evect1(1),evect2(1);evect1(2),evect2(2)];
a1_a2=inv(temp_matrix1)*[vm1_t0;vm2_t0];
a1=a1_a2(1);
a2=a1_a2(2);
```

```
temp_matrix2=[evect1(1)*sqrt(eval1),evect2(1)*sqrt(eval2);evect1(2)*sqrt(eval
1),evect2(2)*sqrt(eval2)];

b1_b2=inv(temp_matrix2)*[am1_t0;am2_t0];
b1=b1_b2(1);
b2=b1_b2(2);


%%% BELOW THIS LINE MIGHT WANT TO CHANGE!!!!

if normal_mode==1
    %a1=0; b1=0; a2=1; b2=1;      %first mode
     a1=1; b1=1; a2=0; b2=0;      %second mode
end

% Now plug these into our equations and can plot versus time (t)
t=0:.1:10;

vm1=evect1(1)*(a1*cos(sqrt(eval1)*t)+b1*sin(sqrt(eval1)*t)) +
evect2(1)*(a2*cos(sqrt(eval2)*t)+b2*sin(sqrt(eval2)*t));

vm2=evect1(2)*(a1*cos(sqrt(eval1)*t)+b1*sin(sqrt(eval1)*t)) +
evect2(2)*(a2*cos(sqrt(eval2)*t)+b2*sin(sqrt(eval2)*t));

%This was the analytical solution FOR PARTICULAR SYSTEM PARAMETERS
vm1_class=sin(t)+sin(sqrt(6)*t);
vm2_class=2*sin(t)-0.5*sin(sqrt(6)*t);

% PLOT TO COMPARE THE SOLUTIONS ('Class solution' for given BCs)_
plot(t,vm1,t,vm2,t,vm1_class,'x',t,vm2_class,'o')
legend('vm1 - Matlab', 'vm2 - Matlab', 'vm1 - Class', 'vm2 - class');
```