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A fractional-order hyperchaotic system and its synchronization

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ABSTRACT

In this paper a novel fractional-order hyperchaotic system is proposed. The chaotic properties of the system in phase portraits are analyzed by using linear transfer function approximation of the fractional-order integrator block. Furthermore, synchronization between two fractional-order systems is achieved by utilizing a single-variable feedback method. Simulation results show that our scheme can not only make the two systems synchronized, but also let them remain in chaotic states.

1. Introduction

In recent years, study on the dynamics of fractional-order differential systems has attracted interest of many researchers. It is demonstrated that some fractional-order differential systems behave chaotically or hyperchaotically, such as the fractional-order Chua's system, the fractional Rossler system, the fractional modified Duffing system and Chen system [1–4]. Recently, Ge et al. also reported chaos in fractional-order van der Pol system and damped Mathieu system [5–7].

To discuss fractional chaotic systems, we usually need to solve fractional-order differential equations. For the fractional differential operator, there are two commonly used definitions: Frünwald–Letnikov (GL) definition and Riemann–Liouville (RL) definition. The latter definition of a derivative of fractional-order α is described by [8]:

$$\frac{\mathrm{d}^\alpha x}{\mathrm{d}t^\alpha} = \frac{1}{\Gamma(n-\alpha)}\frac{\mathrm{d}^n}{\mathrm{d}t^n}\int_0^t (t-\tau)^{n-\alpha-1}x(\tau)\mathrm{d}\tau, \quad n-1 \leqslant \alpha < n \tag{1}$$

where $\Gamma(\cdot)$ is the gamma function.

Ref. [9] presented a numerical comparison between two methods, namely the variational iteration method and the Adomian decomposition method, as well as a conventional fractional difference method for solving linear differential equations of fractional-order. Ref. [10] applied the variational iteration method to nonlinear differential equations of fractional-order. In addition, transfer function approximation in the frequency domain is also frequently used to solve fractional-order differential equations [1,11,12].

Meanwhile, chaotic applications in physics and engineering have caught much attention. A challenging problem is the control and synchronization of chaotic systems. Besides the classical PC synchronization approach presented by Pecora and Carroll [13], there are several other control and synchronization schemes, such as linear or nonlinear state feedback methods [14], a chaotic parameter slight perturbation method, a forced pulse disturbance method, and others. There have been many prior studies that addressed chaos control and synchronization methods in non-fractional-order systems [15–22]. For example, Ref. [15] designed an extended backstepping sliding mode controller, while Ref.[16] presented a fuzzy sliding mode control based method for chaos synchronization. Recently, chaos synchronization problems in fractional-order systems are being widely investigated [23–29]. In Ref. [23], the synchronization of two fractional Lü systems is studied.

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Other examples include Ref. [24] for the chaotic fractional-order unified systems, Ref. [25] for a class of fractional-order chaotic systems, and Ref. [26] for the fractional rational mechanical systems. In this paper we also investigate the synchronization of a new fractional-order hyperchaotic system.

The rest of the paper is organized as follows: In Section 2, a fractional-order hyperchaotic system is presented, and its dynamics is discussed by phase portraits. In Section 3, synchronization between two such hyperchaotic systems is achieved by a single state variable feedback method. Finally, in Section 4, conclusions are drawn.

2. A fractional-order hyperchaotic system

Before we discuss our fractional-order system, a 4D integral-order hyperchaotic system [30] is given by Eq. (2), and its phase portraits are shown in Figs. 1 and 2, respectively. Specifically, Fig. 1 shows the three dimensional (3D) phase portrait of the integral hyperchaotic system, which represents the $x-y-z$ space projection of the hyperchaotic attractor. Fig. 2 depicts the two-dimensional (2D) phase portraits of the system, where Fig. 2a–f represent the $x-y$, $x-z$, $x-w$, $y-z$, $y-w$, $z-w$ plane projections of the phase trajectory, respectively.

$$\begin{cases} \frac{dx}{dt} = ax - y \\ \frac{dy}{dt} = x - yz^2 \\ \frac{dz}{dt} = -b_1 y - b_2 z - b_3 w \\ \frac{dw}{dt} = z + cw \end{cases} \tag{2}$$

where $a = 0.56$, $b_1 = 1.0$, $b_2 = 1.0$, $b_3 = 6.0$, $c = 0.8$.

Based on the above descriptions, we modify the derivative operator in Eq. (2) to be with respect to a fractional-order α. Thus Eq. (2) is converted to Eq. (3):

$$\begin{cases} \frac{d^\alpha x}{dt^\alpha} = ax - y \\ \frac{d^\alpha y}{dt^\alpha} = x - yz^2 \\ \frac{d^\alpha z}{dt^\alpha} = -b_1 y - b_2 z - b_3 w \\ \frac{d^\alpha w}{dt^\alpha} = z + cw \end{cases} \tag{3}$$

In this paper, the following simulations are all performed by using $\alpha = 0.95$.

Assume that the initial conditions are zero. The fractional derivative operator of order α can be represented by the Laplace transform [1]:

$$L\left\{\frac{df^\alpha(t)}{dt^\alpha}\right\} = S^\alpha L\{f(t)\} \tag{4}$$

Basically, the idea is to approximate the system behavior in the frequency domain. With any nonzero initial condition the function will have a singularity at time zero, but this is not necessarily the problem unless the initial value is not infinity [1]. The transfer function is:

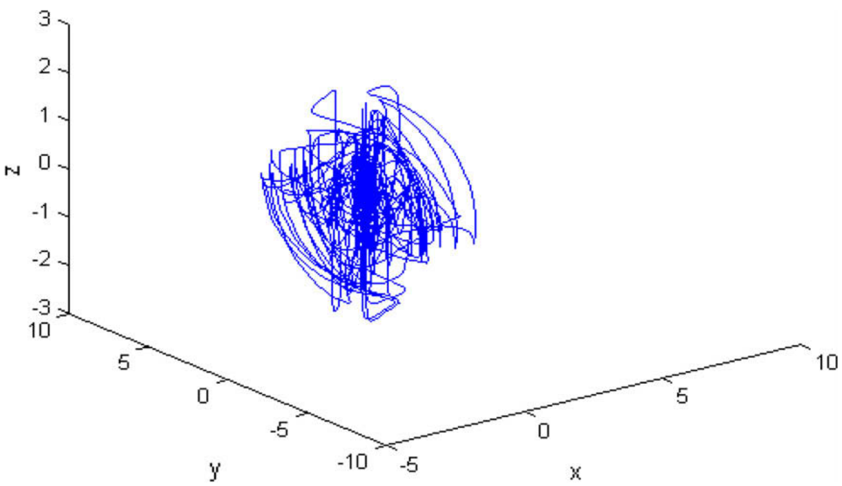

Fig. 1. 3D phase portrait of an integral-order hyperchaotic system described in (2).

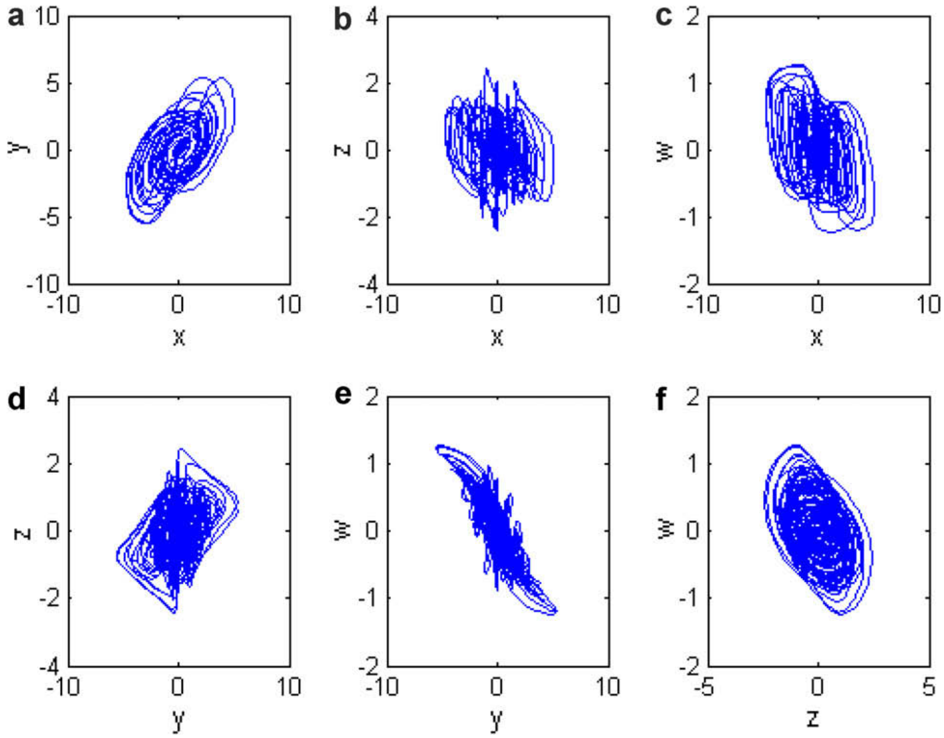

Fig. 2. 2D phase portraits of the integral-order hyperchaotic system described in (2).

$$F(S) = \frac{1}{S^{\alpha}} \tag{5}$$

Here we select following the transfer function approximation method presented in Ref. [3]:

$$\frac{1}{S^{0.95}} \approx \frac{1.281S^2 + 18.6004S + 2.0833}{S^3 + 18.4738S^2 + 2.6574S + 0.003} \tag{6}$$

By simulations we have obtained the 3D and 2D phase portraits of the fractional-order system, as shown in Figs. 3 and 4, respectively. These figures clear show that the fractional-order hyperchaotic system exhibits chaotic behaviors.

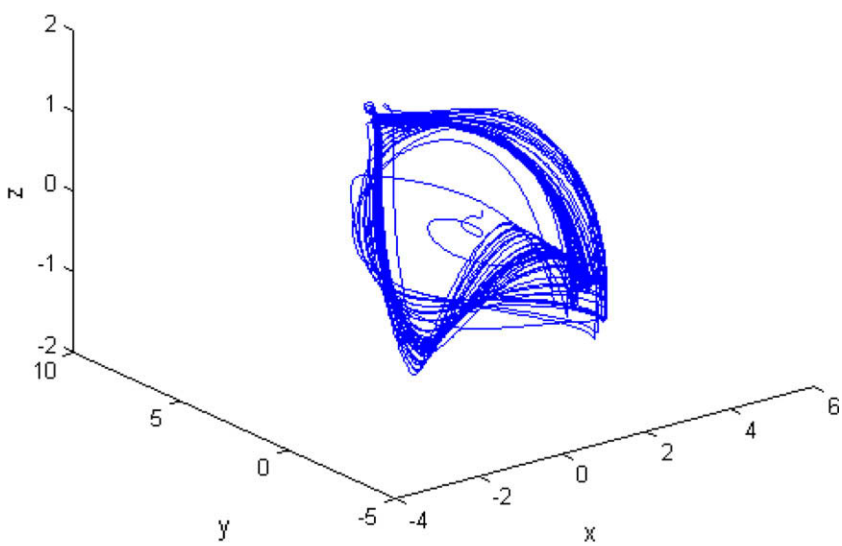

Fig. 3. 3D phase portrait of the fractional-order hyperchaotic system in Eq. (3).

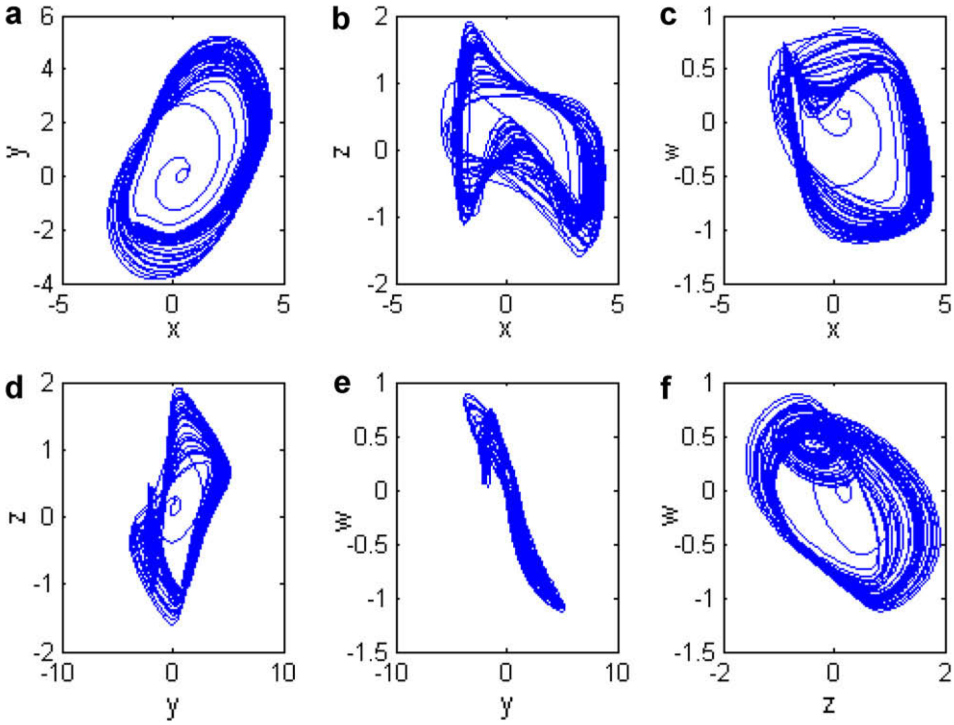

Fig. 4. 2D phase portraits of the fractional-order hyperchaotic system in Eq. (3).

3. Synchronization between two hyperchaotic systems

Consider another hyperchaotic system, which is described by:

$$\begin{cases} \frac{d^\alpha x'}{dt^\alpha} = ax' - y' \\ \frac{d^\alpha y'}{dt^\alpha} = x' - y' \cdot z'^2 \\ \frac{d^\alpha z'}{dt^\alpha} = -b_1 y' - b_2 z' - b_3 w' \\ \frac{d^\alpha w'}{dt^\alpha} = z' + cw' \end{cases} \tag{7}$$

Suppose that Eqs. (3) and (7) denote two independent systems with the same parameters but different initial conditions: $(x_0, y_0, z_0, w_0) = (0.7, 0.1, 0.3, 0.1)$, and $(x'_0, y'_0, z'_0, w'_0) = (1.2, 0.6, 0.8, 0.5)$. The time waveforms of the two systems are shown in Fig. 5. From this figure we can see that the two systems remain in independent hyperchaotic states. However, if one of them is regarded as the driving system, the other as the response system, and a feedback control $u(t)$ is applied in one of the response state equations, then the synchronization of the two systems can proceed as described in Eq. (8), where $u(t) = x - x'$, (x, y, z, w) denotes the variables of the response system, and (x', y', z', w') denotes the variables of the driving system.

$$\begin{cases} \frac{d^\alpha x}{dt^\alpha} = ax - y - u(t) \\ \frac{d^\alpha y}{dt^\alpha} = x - yz^2 \\ \frac{d^\alpha z}{dt^\alpha} = -b_1 y - b_2 z - b_3 w \\ \frac{d^\alpha w}{dt^\alpha} = z + cw \\ \frac{d^\alpha x'}{dt^\alpha} = ax' - y' \\ \frac{d^\alpha y'}{dt^\alpha} = x' - y' \cdot z'^2 \\ \frac{d^\alpha z'}{dt^\alpha} = -b_1 y' - b_2 z' - b_3 w' \\ \frac{d^\alpha w'}{dt^\alpha} = z' + cw' \end{cases} \tag{8}$$

The synchronization errors between the two systems obtained by simulation are shown in Fig. 6. Specifically, Fig. 6a–d denote the errors of $x(t) - x'(t)$, $y(t) - y'(t)$, $z(t) - z'(t)$, $w(t) - w'(t)$, respectively.

Fig. 6 demonstrates that the two systems are synchronized after some time. The phase portraits of the driving and response systems are shown in Figs. 7–9, respectively.

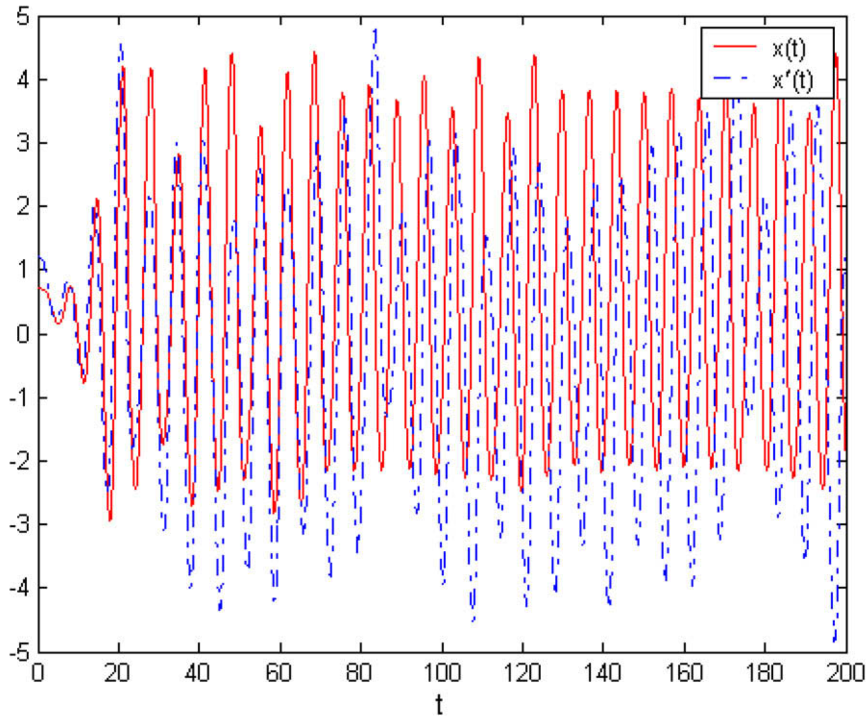

Fig. 5. The time waveform $x(t)$ and $x'(t)$ of the two hyperchaotic systems with different initial conditions, where $(x_0, y_0, z_0, w_0) = (0.7, 0.1, 0.3, 0.1)$, and $(x'_0, y'_0, z'_0, w'_0) = (1.2, 0.6, 0.8, 0.5)$.

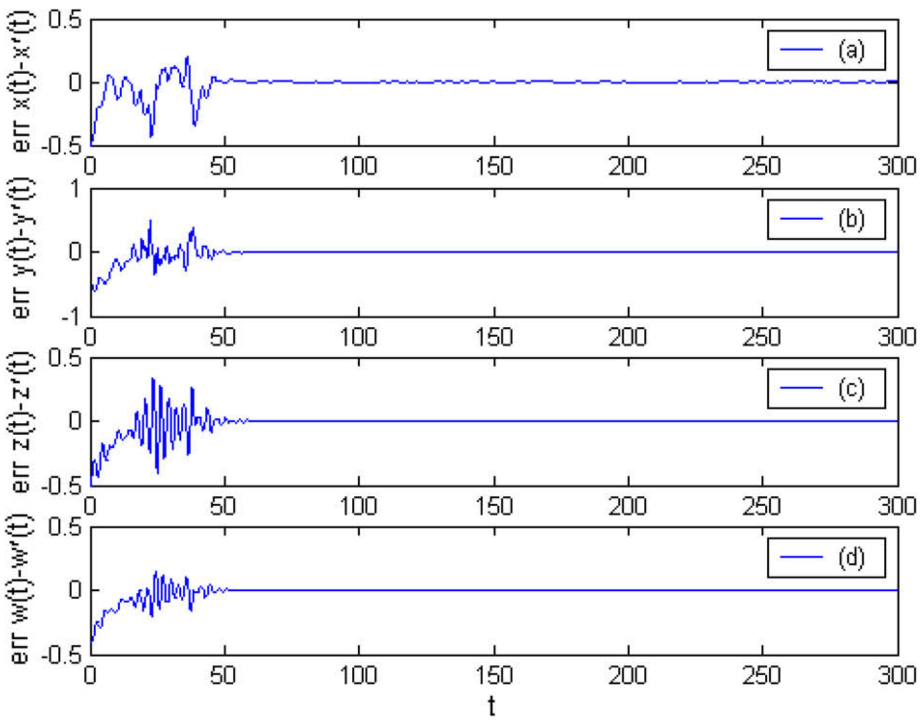

Fig. 6. The synchronization error functions of four state variables versus time *t*.

Figs. 7 and 8 depict the 2D portraits of the driving and the response systems, respectively. Fig. 9a and b depict the 3D phase portraits of the driving and response systems, respectively. From these figures, we see that the two systems remain

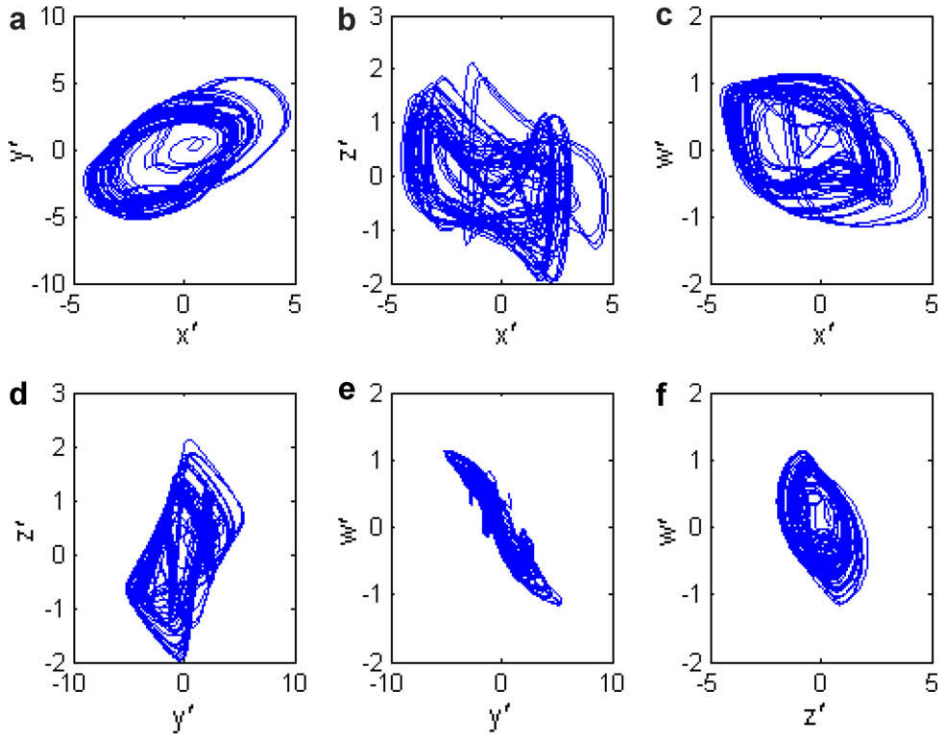

Fig. 7. 2D phase portraits of the driving system.

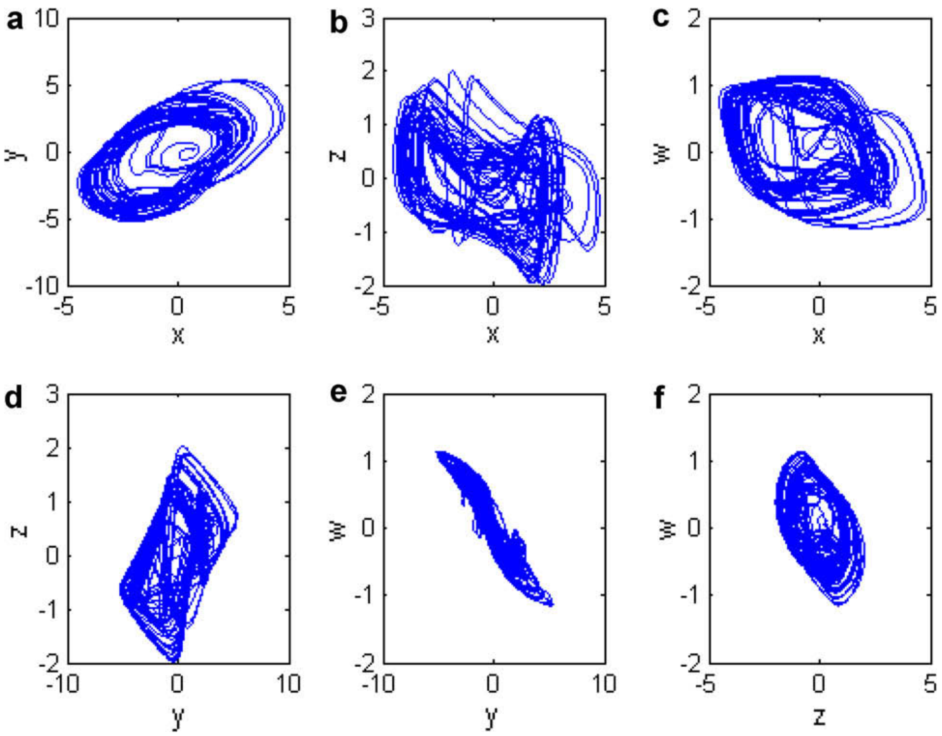

Fig. 8. 2D phase portraits of the response system.

in chaotic states. This is different from Ref. [26]. The synchronization of the two systems is implemented in Ref. [26], where the systems after synchronization are no longer chaotic, but exhibit some periodic dynamic behaviors. Although some other

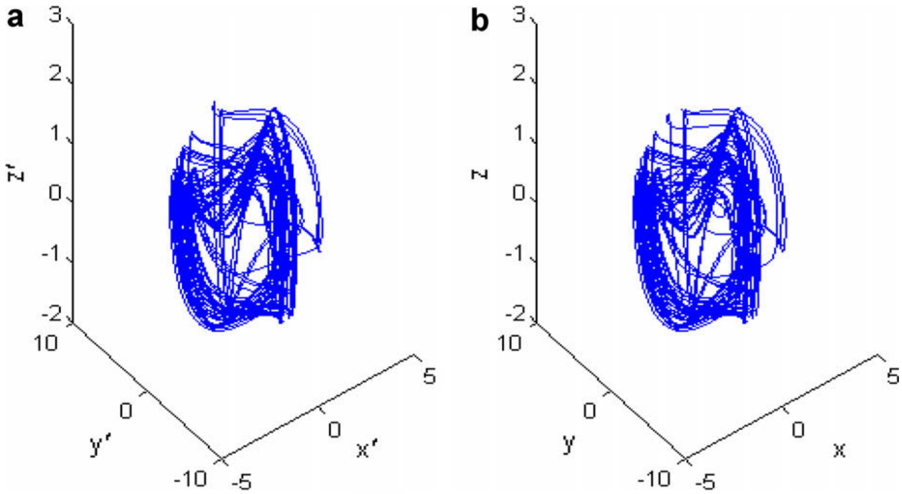

Fig. 9. 3D phase portraits of the driving and response systems.

studies [25,27–29] achieved the synchronization in chaotic states, they are limited to those fractional-order systems with only three state variables. There are few studies on the synchronization of fractional-order hyperchaotic systems [31,32].

4. Conclusion

In this paper a fractional-order hyperchaotic system with order 3.8 ($\alpha = 0.95$) is proposed. Its dynamical behaviors are studied. Moreover, synchronization between two such hyperchaotic systems has been achieved via a simple feedback control. Simulation results show that the two systems can maintain complex chaotic states after synchronization.

Acknowledgements

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