Distributed Estimation of Gauss–Markov Random Fields With One-Bit Quantized Data

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***Abstract*—We consider the problem of distributed estimation of a Gauss–Markov random field using a wireless sensor network (WSN), where due to the stringent power and communication constraints, each sensor has to quantize its data before transmission. In this case, the convergence of conventional iterative matrix-splitting algorithms is hindered by the quantization errors. To address this issue, we propose a one-bit adaptive quantization approach which leads to decaying quantization errors. Numerical results show that even with one bit quantization, the proposed approach achieves a superior mean square deviation performance (with respect to the global linear minimum mean-square error estimate) within a moderate number of iterations.**

***Index Terms*—Adaptive quantization (AQ), distributed estimation, Gauss–Markov random fields (GMRFs).**

I. Introduction

DISTRIBUTED estimation is a fundamental problem arising from a wide range of sensor network applications, such as environment monitoring and battlefield surveillance. So far a lot of existing works [1]–[5] studied the distributed estimation problem by modeling the phenomena of interest as a common scalar parameter. In this paper, we consider another type of problems that model the underlying phenomena as a random field. A random field is a generalization of a single random variable, where the underlying parameter is no longer a common scalar, but instead a stochastic process on a multidimensional vector space or a manifold. Specifically, we focus on an important class of random fields: Gauss–Markov random fields (GMRFs), which have been widely used to model spatially distributed phenomena such as temperature, wind speed, and concentration of some chemical material in many large-scale estimation problems. Distributed estimation of GMRFs has been investigated in [6]–[11]. Most schemes [8]–[11] are parallel, iterative and involve communication only among neighboring sensors. In [8], [9], the authors studied how to accelerate the convergence rate of the matrix-splitting iterative algorithms by exploiting embedded trees [8] or embedded graphs [9]. Robust design against temporary communication failures such as failing links and sleeping nodes was examined in [9], [10]. Fixing convergence in cases where the spectral radius condition is not met was studied in [11]. Nevertheless, all of them assume that data are exchanged among neighboring sensors without distortion, which may not be practical due to stringent power and bandwidth constraints. It is therefore meaningful to examine how the quantization errors affect those iterative algorithms and to design a robust scheme for distributed random field estimation in the presence of quantization errors.

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II. Problem Formulation

Suppose that we have N spatially distributed sensors. Each sensor makes a noisy scalar measurement of a GMRF:

$$d_n = \theta_n + w_n \quad n = 1, \ldots, N \tag{1}$$

where θ_n is the true value of the random field at sensor n and is assumed constant over time, d_n and w_n denote the sensor measurement and noise, respectively. We assume that the random vector $\boldsymbol{\theta} \triangleq [\theta_1 \ \theta_2 \ \ldots \ \theta_N]^T$ has zero mean with covariance matrix $\mathbf{R}_\theta$, and the measurement noise $\mathbf{w} \triangleq [w_1 \ w_2 \ \ldots \ w_N]^T$ is independent of $\boldsymbol{\theta}$ with zero mean and diagonal covariance matrix $\mathbf{R}_w$. The global linear minimum mean-square error (LMMSE) estimate of the random field is given by [9]

$$\hat{\boldsymbol{\theta}} = \mathbf{R}^{-1}\mathbf{R}_w^{-1}\mathbf{d} \tag{2}$$

where $\mathbf{d} \triangleq [d_1 \ d_2 \ \ldots \ d_N]^T$, $\hat{\boldsymbol{\theta}} \triangleq [\hat{\theta}_1 \ \hat{\theta}_2 \ \ldots \ \hat{\theta}_N]^T$, and $\mathbf{R} \triangleq \mathbf{R}_\theta^{-1} + \mathbf{R}_w^{-1}$.

The GMRFs are usually parameterized by the covariance matrix $\mathbf{R}_\theta$. An alternate natural parameterization for GMRFs is the so-called precision matrix $\mathbf{Q}$, which is defined as the inverse of the covariance matrix $\mathbf{R}_\theta$, i.e., $\mathbf{Q} \triangleq \mathbf{R}_\theta^{-1}$. It is shown in [8] that $\mathbf{Q}$ is a sparse matrix with its entries $q_{i,j} = 0$ if the variables θ_i and θ_j, conditioned on all other hidden variables $\{\theta_n\}_{n \in V \backslash i,j}$, are independent, where $V \backslash i,j$ denotes the set of nodes obtained by excluding nodes i and j. This property enables the GMRFs to be conveniently represented by an undirected and locally connected graph $G = (V, E)$, where the vertices $V = \{1, 2, \ldots, N\}$ correspond to the random variables $\{\theta_n\}$, and the edge structure $E = \{(i,j) | i,j \in V\}$ specifies the conditional independence properties among the variables $\{\theta_n\}$. We assume that each sensor, say sensor i, can obtain the knowledge of $\{q_{i,j}\}, \forall j$ from a pre-specified model of the precision matrix $\mathbf{Q}$ [8], [9].

III. Matrix-Splitting Algorithms

Matrix splitting [8], [9] is an iterative approach to solve the linear system (2). Let $\mathbf{y} \triangleq \mathbf{R}_w^{-1}\mathbf{d}$, then we are faced with

$$\hat{\boldsymbol{\theta}} = \mathbf{R}^{-1}\mathbf{y}. \tag{3}$$

To solve (3), $\mathbf{R}$ is split into $\mathbf{R} = \mathbf{J} - \mathbf{K}$. The LMMSE estimate $\hat{\boldsymbol{\theta}}$ can be found by the following two-step recursion (see [8], [9] and references therein for more details):

$$\begin{aligned}\mathbf{z}(t) &= \mathbf{y} + \mathbf{K}\hat{\boldsymbol{\theta}}(t-1)\\ \hat{\boldsymbol{\theta}}(t) &= \mathbf{J}^{-1}\mathbf{z}(t).\end{aligned} \tag{4}$$

Note that $\mathbf{R}$ is a sparse matrix with nonzero entries only at $(i,j) \in E$. Therefore it is possible to seek a pair $(\mathbf{J},\mathbf{K})$ for which the two-step recursion involves communications only among neighboring sensors. For example, we can let $\mathbf{J}$ equal to the diagonal of $\mathbf{R}$, which reduces to the well-known Jacobi algorithm. More sophisticated splitting design which leads to a faster convergence rate has also been proposed in other works, e.g., [9]. A necessary and sufficient condition for the above recursion to converge is $\rho(\mathbf{J}^{-1}\mathbf{K}) < 1$, where $\rho(\cdot)$ denotes the spectral radius of a matrix.

IV. Convergence Analysis

In practice, quantization has to be carried out before transmission. In this case, we modify the recursion (4) as

$$\begin{aligned}\mathbf{z}(t) &= \mathbf{y} + \mathbf{K}Q(\hat{\boldsymbol{\theta}}(t-1))\\ \hat{\boldsymbol{\theta}}(t) &= \mathbf{J}^{-1}Q(\mathbf{z}(t))\end{aligned} \tag{5}$$

where $Q(\mathbf{x})$ denotes the quantized vector by carrying out quantization for each entry of $\mathbf{x}$. From (5), we see that only the quantized version of the data, i.e., $Q(\hat{\theta}_i(t-1))$ and $Q(z_i(t))$, are exchanged among neighboring sensors (y_i need not be exchanged among sensors). We write $Q(\hat{\boldsymbol{\theta}}(t-1)) = \hat{\boldsymbol{\theta}}(t-1) + \mathbf{v}(t-1)$ and $Q(\mathbf{z}(t)) = \mathbf{z}(t) + \mathbf{u}(t)$, where $\mathbf{v}(t) \triangleq [v_1(t)\ v_2(t)\ \ldots\ v_N(t)]^T$, $\mathbf{u}(t) \triangleq [u_1(t)\ u_2(t)\ \ldots\ u_N(t)]^T$, and $v_n(t)$ and $u_n(t)$ denote the quantization errors of sensor n introduced at iteration t. From (5), the state $\hat{\boldsymbol{\theta}}(t)$ can be expressed in terms of the initial state $\hat{\boldsymbol{\theta}}(0)$ as

$$\begin{aligned}\hat{\boldsymbol{\theta}}(t) = (\mathbf{J}^{-1}\mathbf{K})^t\hat{\boldsymbol{\theta}}(0) &+ \sum_{i=0}^{t-1}(\mathbf{J}^{-1}\mathbf{K})^i\mathbf{J}^{-1}\mathbf{y}\\ &+ \sum_{i=0}^{t-1}(\mathbf{J}^{-1}\mathbf{K})^{t-i}\mathbf{v}(i) + \sum_{i=1}^{t}(\mathbf{J}^{-1}\mathbf{K})^{t-i}\mathbf{J}^{-1}\mathbf{u}(i).\end{aligned} \tag{6}$$

As time evolves, the first term vanishes since $\rho(\mathbf{J}^{-1}\mathbf{K}) < 1$; the second term approaches the desired LMMSE estimate [8]

$$\sum_{i=0}^{t-1}(\mathbf{J}^{-1}\mathbf{K})^i\mathbf{J}^{-1}\mathbf{y} \xrightarrow{t\to\infty} (\mathbf{J}-\mathbf{K})^{-1}\mathbf{y} = \mathbf{R}^{-1}\mathbf{y}. \tag{7}$$

However, the last two terms induced by the quantization errors still remain, which leads to a system divergence. Nevertheless, a closer examination reveals that, for a specific i, the quantization error components $\mathbf{v}(i)$ and $\mathbf{u}(i)$ incurred at iteration i will eventually vanish as $t \to \infty$. This observation implies that a convergence may be achieved if the quantization errors decay to zero with time. By modeling the quantization errors as random variables, we have the following result.

Proposition 1: Suppose that $(\mathbf{J},\mathbf{K})$ satisfies the condition $\|\mathbf{J}^{-1}\mathbf{K}\|_2 < 1$, and the quantization errors $\{v_n(t)\}$ and $\{u_n(t)\}$ are zero-mean, spatially and temporally uncorrelated random variables. If the variances of the quantization errors decay over time, i.e., $E[v_n^2(t)] \to 0$, $E[u_n^2(t)] \to 0$ as $t \to \infty\ \forall n$, then the recursion (5) converges to the LMMSE estimate in a mean-square sense.

Proof: See Appendix A. ∎

Remarks: Proposition 1 requires the matrices $\mathbf{J}$ and $\mathbf{K}$ satisfy $\|\mathbf{J}^{-1}\mathbf{K}\|_2 < 1$, which is a condition more restrictive than $\rho(\mathbf{J}^{-1}\mathbf{K}) < 1$. It was shown in [12], [13] that the quantization errors can be modeled as zero-mean spatially and temporally uncorrelated random variables under a sufficient condition that the joint characteristic function of the input messages is bandlimited. Since the input messages are always bounded, the joint characteristic function of the input messages cannot be exactly bandlimited in practice and the above assumptions hold valid only in an approximate sense. On the other hand, introducing dithering to the system can also lead to independent quantization errors, as shown in [5], [13]. Its application to our context is an interesting problem worthy of future investigation.

The theoretical result of Proposition 1 inspires us to propose a one-bit adaptive quantization (AQ) scheme which can potentially achieve decaying quantization errors by exploiting the temporal correlation among successive states. A similar approach was developed in [14]. Nevertheless, the approach was studied within a decentralized framework (with a fusion center) and the objective is to estimate a common parameter through a sequential sensor update.

V. Proposed Approach

The AQ scheme involves an encoding and decoding process. The received encoded data has to be decoded before it is applied to the recursive update (5). Hence the quantized data we discussed in (5) correspond to the data decoded at the receiver, but not the encoded data at the transmitter.

Let us consider encoding first. For each sensor, say sensor n, it, firstly, uses two globally specified parameters: an initial threshold τ, and an initial quantization step-size Δ, to generate its one-bit encoded data of the first two iterations:

$$\begin{aligned}b_n(0) &= \mathrm{sgn}(x_n(0) - \tau)\\ b_n(1) &= \mathrm{sgn}(x_n(1) - \tau_n(1))\end{aligned} \tag{8}$$

where $\mathrm{sgn}\{x\} = -1$ if $x \le 0$, otherwise $\mathrm{sgn}\{x\} = 1$, $\tau_n(1) = \tau + b_n(0)\Delta$, $x_n(t)$ denotes the raw data (can be $\hat{\theta}_n(t)$ or $z_n(t)$ in (5)), $b_n(t)$ denotes the encoded data of sensor n at iteration t, and $\tau_n(t)$ is the corresponding threshold used for quantization. At iteration $t \ge 1$, sensor n computes its threshold by performing accumulation of the previous bits, weighted by a variable step-size $\Delta_n(t)$:

$$\tau_n(t+1) = \tau_n(t) + b_n(t)\Delta_n(t) \tag{9}$$

where $\Delta_n(t)$ evolves using the following dynamic model

$$\Delta_n(t) = \begin{cases} \Delta_n(t-1)K^{b_n(t)b_n(t-1)}, & \text{if } \Delta_n(t) < \lambda \\ \lambda, & \text{otherwise} \end{cases} \quad (10)$$

where $K > 1$ is a constant, $\Delta_n(0) = \Delta$, and λ is a parameter of user choice in order to prevent a too large step-size. Then sensor n uses $\tau_n(t+1)$ as a threshold to generate its encoded data at iteration $t+1$:

$$b_n(t+1) = \text{sgn}(x_n(t+1) - \tau_n(t+1)). \quad (11)$$

The decoding of the AQ scheme is simple and described as follows. Suppose sensor m is one of the neighboring sensors of sensor n. After receiving the encoded data $b_n(t)$ from sensor n, sensor m recovers the quantization threshold $\tau_n(t+1)$ and uses it as the decoded output data for the recursion, i.e., $Q(x_n(t)) = \tau_n(t+1)$. The reconstruction of $\tau_n(t+1)$ can be easily inferred from the received encoded data $\{b_n(i)\}_{i=0}^{t}$ in a recursive manner by using (9)–(10).

A. Discussions

We can recognize that the above process is reminiscent of the Delta modulation with variable step-size, but implemented in a distributed fashion. The key idea of AQ is to adjust the step-size based on two successive encoded bits. When successive encoded bits have identical signs, with a high probability it is still in the catch-up phase and the step-size is increased to speed up the process [c.f (10)]. On the other hand, alternating signs between successive bits indicate that the quantized data are oscillating around the waveform, in which case the step-size is decreased [c.f (10)] to provide a finer quantization.

We provide a heuristic discussion (not a rigorous proof) to show how the AQ approach induces decaying quantization errors in our distributed context. Note that the recursive estimate $\hat{\boldsymbol{\theta}}(t)$ of (6), on its steady state, is the desired fix-point solution $\hat{\boldsymbol{\theta}}$ plus two noise items. Therefore, for each sensor, we can write $\hat{\theta}_n(t) = \hat{\theta}_n + e_n(t)$, where $e_n(t)$ denotes the additive noise introduced by the quantization errors. To simplify our analysis, we idealize the recursive process into a two-phase recursion. In the first phase, we suppose that the dynamic range of the noise $e_n(t)$ is constant and examine the evolution of the quantization step-size. Assume that the noise $e_n(t)$, $\forall n$ is bounded by $[-B, B]$ over a short duration. It can be shown that the adaptive quantization adjustment will lead to a quantization step-size $\Delta_n(t)$ smaller than the dynamic range $2B$, otherwise it will generate successive alternating signs to reduce the quantization step-size. In the second phase, suppose now the quantization step-size is fixed and see how it affects the additive noise $e_n(t)$, $\forall n$. It can be verified from (6) that when $\rho(\mathbf{J}^{-1}\mathbf{K})$ is smaller than a certain value, a fixed quantization step-size Δ will result in noise $e_n(t)$, $\forall n$ whose practical dynamic range is smaller than Δ (it is, however, difficult to check whether or not the bounded spectral radius condition is satisfied in distributed setting). In this case, the two phases form a negative feedback leading to attenuating quantization errors.

The AQ scheme can be easily implemented since it only involves very simple algebraic recursive calculations. Also, for each sensor, no knowledge of the global topology is required. Each sensor only needs the following globally pre-specified parameters: an initial threshold τ, an initial quantization step-size Δ, and parameters K, λ used for quantization step-size update (10). The AQ approach is not quite sensitive to the initial choice of τ and Δ because these two parameters are adaptively adjusted during the iterative process. On the other hand, the parameters λ and K are kept constant throughout the iterations and hence are more critical. Our experiments suggest that in order to prevent a too large step-size, a small $K \in (1, 2]$ and $\lambda \in (0, 2]$ are preferable in most cases. From (9)–(11), we see that for both encoding and decoding, only current data and information from past two states are required. Hence only a small amount of memory resource is required to store this information. Note that each sensor only needs to encode its own data and, at the same time, to decode the data received from its neighboring sensors.

VI. Simulation Results

We consider $N = 50$ sensors placed uniformly at random on a 2-D unit area. The observations of the sensors are generated according to model (1), where $\{w_n\}$ have zero mean and covariance matrix $\mathbf{R}_w = \sigma_w^2\mathbf{I}$, the random variables $\{\theta_n\}$ have conditional correlation coefficients given by

$$\text{Corr}(\theta_i, \theta_j | \theta_{V\setminus i,j}) = \begin{cases} \frac{1}{\sqrt{r_i r_j}}, & \text{if } (i,j) \in E \\ 0, & \text{otherwise} \end{cases} \quad (12)$$

where $(i,j) \in E$ if and only if the Euclidean distance between nodes i and j is smaller than $R = \sqrt{\log N/N}$ to ensure that the graph is connected with a high probability [15]; r_i denotes the degree of node i, i.e., the number of connections it has to other nodes. The entries of the precision matrix $\mathbf{Q}$, as shown in [9], are related to the partial correlation coefficients by

$$\text{Corr}(\theta_i, \theta_j | \theta_{V\setminus i,j}) = \frac{-q_{i,j}}{\sqrt{q_{i,i}q_{j,j}}} \quad (13)$$

where the diagonal elements of $\mathbf{Q}$ is chosen to be $q_{i,i} = r_i + c$, with $c = 0.1$ to ensure $\mathbf{Q}$ is positive definite. The off-diagonal entries of $\mathbf{Q}$ can be computed from (12) and (13). The splitting matrix $\mathbf{J}$ is chosen equal to the diagonal of $\mathbf{R}$.

To illustrate the effectiveness of the AQ approach, we compare it with a uniform quantization (UQ) scheme. The performance is measured by a normalized mean-square error (NMSE) between the recursion state and the desired global LMMSE estimate (2), i.e., $(1/N)E[\|\hat{\boldsymbol{\theta}}(t) - \hat{\boldsymbol{\theta}}\|_2^2]$. In addition, the NMSE between the recursion state and the true random field, $(1/N)E[\|\hat{\boldsymbol{\theta}}(t) - \boldsymbol{\theta}\|_2^2]$, is included. Results are averaged over 500 Monte Carlo runs, with the graph and the initial state independently generated for each run. Fig. 1 shows the NMSEs of the AQ approach and the UQ scheme with $q = 1, 5, 9$, respectively, where q denotes the number of bits used for quantization. The no quantization (NQ) scheme is also included for comparison, which provides a benchmark on the achievable performance of all rate-constrained methods. From Fig. 1(a), we see that due to the non-decaying quantization errors, the UQ scheme does not converge and the NMSE is kept at a constant level after a certain point, even for $q = 9$. In contrast, the AQ approach demonstrates a consistent performance improvement and presents a clear advantage over the UQ scheme. From Fig. 1(b), we see that the AQ approach can offer a level of

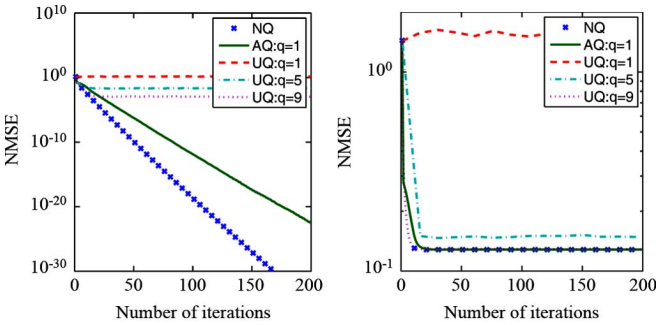

Fig. 1. (a) NMSEs between $\hat{\boldsymbol{\theta}}(t)$ and the global estimate (2); (b) NMSEs between $\hat{\boldsymbol{\theta}}(t)$ and the true random field.

performance even comparable to the algorithm with no quantization, with considerable bandwidth/power savings.

VII. Conclusion

The problem of distributed estimation of GMRFs with quantized data was studied. We show that the convergence can still be achieved in the presence of quantization errors if they satisfy a certain decaying condition. An adaptive quantization approach was proposed. Numerical results are illustrated to show the effectiveness of the proposed approach.

Appendix
Proof of Proposition 1

Our objective is to show that

$$E[(\hat{\boldsymbol{\theta}}(t)-\hat{\boldsymbol{\theta}})^T(\hat{\boldsymbol{\theta}}(t)-\hat{\boldsymbol{\theta}})]\rightarrow 0 \quad as\ t\rightarrow\infty \tag{14}$$

where $\hat{\boldsymbol{\theta}}(t)$ is defined in (6). Clearly, as $t\rightarrow\infty$, the first term of (6) vanishes and the second term approaches to the fixed-point solution $\hat{\boldsymbol{\theta}}$. Therefore we only need to prove that

$$\begin{aligned}&E[(\mathbf{g}_1(t)+\mathbf{g}_2(t))^T(\mathbf{g}_1(t)+\mathbf{g}_2(t))]\\&\quad=E[\mathbf{g}_1(t)^T\mathbf{g}_1(t)]+E[\mathbf{g}_2(t)^T\mathbf{g}_2(t)]\rightarrow 0 \quad as\ t\rightarrow\infty\end{aligned} \tag{15}$$

where $\mathbf{g}_1(t)$ and $\mathbf{g}_2(t)$ denote the third and fourth terms on the RHS of (6), respectively. Considering $\mathbf{g}_1(t)$, we have

$$\begin{aligned}E[\mathbf{g}_1(t)^T\mathbf{g}_1(t)]&=\mathrm{tr}\{E[\mathbf{g}_1(t)\mathbf{g}_1(t)^T]\}\\&\overset{(a)}{\leq}\sum_{i=0}^{t-1}\sigma_v^2(i)\|(\mathbf{J}^{-1}\mathbf{K})^{t-i}\|_{\mathrm{F}}^2\end{aligned} \tag{16}$$

where $\|\cdot\|_{\mathrm{F}}$ denotes Frobenius norm, (a) follows from the fact that $\{\mathbf{v}(t)\}$ is spatially and temporally uncorrelated, and $\sigma_v^2(t)$ denotes the largest diagonal element of the auto-covariance matrix $E[\mathbf{v}(t)\mathbf{v}(t)^T]$. Since the quantization errors converge to zero, there exists t_0 such that for any arbitrary small $\epsilon>0$, $\sigma_v^2(t)<\epsilon$ holds for $t>t_0$. Therefore (16) is upper bounded

$$\begin{aligned}&\sum_{i=0}^{t-1}\sigma_v^2(i)\|(\mathbf{J}^{-1}\mathbf{K})^{t-i}\|_{\mathrm{F}}^2\\&\quad<\sum_{i=0}^{t_0}\sigma_v^2(i)\|(\mathbf{J}^{-1}\mathbf{K})^{t-i}\|_{\mathrm{F}}^2+\epsilon\sum_{i=t_0+1}^{t-1}\|(\mathbf{J}^{-1}\mathbf{K})^{t-i}\|_{\mathrm{F}}^2\end{aligned} \tag{17}$$

where the first term vanishes as $t\rightarrow\infty$ since $\rho(\mathbf{J}^{-1}\mathbf{K})<1$, the second term is bounded by

$$\begin{aligned}\epsilon\sum_{i=t_0+1}^{t-1}\|(\mathbf{J}^{-1}\mathbf{K})^{t-i}\|_{\mathrm{F}}^2&\overset{(a)}{\leq}\epsilon N\sum_{i=t_0+1}^{t-1}\|(\mathbf{J}^{-1}\mathbf{K})^{t-i}\|_2^2\\&\overset{(b)}{<}\epsilon N\frac{\|\mathbf{J}^{-1}\mathbf{K}\|_2^2}{1-\|\mathbf{J}^{-1}\mathbf{K}\|_2^2}\end{aligned} \tag{18}$$

where (a) comes from the norm inequality $\|\mathbf{A}\|_{\mathrm{F}}\leq\sqrt{p}\|\mathbf{A}\|_2$ [16], p is the dimension of the square matrix $\mathbf{A}$; (b) follows from $\|\mathbf{AB}\|_2\leq\|\mathbf{A}\|_2\|\mathbf{B}\|_2$ and the assumption $\|\mathbf{J}^{-1}\mathbf{K}\|_2<1$. We see that as $t\rightarrow\infty$, and $\epsilon\rightarrow 0$, both terms on the right-hand side of (17) will vanish. Similarly, we can get the same result on $\mathbf{g}_2(t)$.

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