Identifying unambiguous frequency pattern for target localisation using frequency diverse array

Jingjing Li✉, Hongbin Li and Shan Ouyang

The beampattern of a conventional frequency diverse array (FDA) with linear frequency pattern (i.e. the centre frequency of each antenna element is linearly increasing) is coupled in range and angle. FDA with random frequency pattern can decouple the range and angle with high probability only when the number of elements is sufficiently large. The ambiguity of frequency patterns that are random permutations of a linear FDA is examined. An eigenvector-based criterion to identify possible ambiguity in target localisation using such arrays is proposed. Numerous results are presented to verify the effectiveness of the proposed ambiguity identification criterion.

Introduction: The frequency diverse array (FDA) has attracted increasing attention in recent years [1–9]. FDA can produce a range–angle-dependent beampattern by employing a small linear frequency increment across the antenna elements. This range–angle-dependent beampattern can be exploited for range-dependent interference suppression [1], range ambiguity resolution [2], and performance enhancement in synthetic aperture radar [3] and moving target indication [4].

However, the beampattern of a conventional FDA with linear frequency pattern is coupled in range and angle dimensions, that is, there might be multiple range–angle pairs that match well with the echo from a point target at a specific spatial location, hence introducing ambiguity in target indication [5]. To overcome this problem, several decoupled methods were introduced. FDA with nonlinear frequency increments based on, e.g. a logarithmic frequency pattern [6], square or cubic frequency pattern [7], can achieve a beampattern with a single maximum at the target location. Meanwhile, the range and angle of the target can be decoupled by using a MIMO technique [8]. Some other methods based on multi-carrier modulation and subarray were also investigated.

Random FDA was proposed in [5] for decoupled target indication with low system complexity. Stochastic characteristics of the beampattern including the mean, variance, and asymptotic distribution were analytically derived. These results are accurate for arrays with a sufficiently large number of elements. However, when the number of elements is small, the random FDA cannot always produce a decoupled range–angle beampattern. In some cases, it may produce a beampattern with multiple ambiguous mainlobes. Therefore, it is of interest to develop some simple criteria to identify unambiguous random frequency patterns for target localisation using small-scale arrays. In this Letter, we consider frequency patterns that are random permutations of a linear FDA with a small number of elements. Since the beampattern cannot be written in a closed form, an eigenvector-based method is proposed to analyse the relationship between the frequency pattern and ambiguity. This leads to a simple criterion that can be used to identify unambiguous frequency patterns for target localisation.

Signal model: Consider a uniform linear array consisting of N elements. The carrier frequency at the nth element is

$$f_n = f_0 + m_n \Delta f, \quad n = 0, 1, \ldots, N-1 \tag{1}$$

where Δf is the frequency increment across the array elements and f_0 is the reference carrier frequency. m_n is the nth element of the $N \times 1$ *frequency pattern* vector $\boldsymbol{m}$ containing a random permutation of the integers 0 to $N-1$.

The transmitted signal of the nth element can be expressed as $s_n(t) = \mathrm{e}^{\mathrm{j}2\pi f_n t}$. We consider the case where the receiver is band limited and the nth element only receives/processes the signal with carrier frequency f_n [9]. Choose the first element as the reference. Thus, the received echo of the nth element is [5]

$$\begin{aligned} y_n(t, r, \theta) &= \beta_0 \, \mathrm{e}^{\mathrm{j}2\pi f_n[t-((2(r-nd\sin\theta))/c)]} \\ &= \beta_0 \, \mathrm{e}^{\mathrm{j}2\pi(f_0+m_n\Delta f)t} \, \mathrm{e}^{-\mathrm{j}2\pi f_0(2r/c)} \, \mathrm{e}^{\mathrm{j}2\pi\left(n((2f_0 d\sin\theta)/c)-m_n(2\Delta f r/c)\right)} \end{aligned} \tag{2}$$

where β_0 stands for the complex reflection amplitude of the target, θ denotes the angle of the target (measured from the normal direction to the target direction), r is the slant range of the target, whereas d is the interspacing of the elements. To avoid the grating lobes in angle dimension of the transmit–receive beampattern, we consider $d \le \lambda_0/4$ [5, 9]. The item $nm_n((2\Delta f d\sin\theta)/c)$ is ignored since $\Delta f \ll f_0$ and $d\sin\theta \ll r$. After coherent demodulation, (2) becomes

$$y_n(r, \theta) = \xi(r)\, \mathrm{e}^{\mathrm{j}\Theta(n, m_n, r, \theta)} \tag{3}$$

where $\xi(r) = \beta_0 \, \mathrm{e}^{-\mathrm{j}2\pi f_0(2r/c)}$ and $\Theta(n,m_n,r,\theta) = 2\pi\left(n((2f_0 d\sin\theta)/c) - m_n(2\Delta f r/c)\right)$.

Since we are interested in the relationship between the frequency pattern and ambiguity in target localisation, we consider the scenario of one target and no noise. Therefore, the received signal can be written in vector form as

$$\boldsymbol{y}(r, \theta) = \boldsymbol{a}(r, \theta)\xi(r) \tag{4}$$

where $\boldsymbol{a}(r, \theta) = [\mathrm{e}^{\mathrm{j}\Theta(0, m_0, r, \theta)}, \ldots, \mathrm{e}^{\mathrm{j}\Theta(N-1, m_{N-1}, r, \theta)}]^{\mathrm{T}}$ and $\boldsymbol{y}(r,\theta) = [y_0(r,\theta), y_1(r,\theta), \ldots, y_{N-1}(r,\theta)]^{\mathrm{T}}$.

Proposed criterion: In conventional FDAs with a linear frequency pattern $m_n = n$. Since the elements of the steering vector $\boldsymbol{a}(r, \theta)$ are a geometric series, the beampattern can be written in a closed form. However, this is impossible for random FDAs. In the following, we introduce an eigenvector-based method to analyse the array response. For notational simplicity, we henceforth use Θ_n instead of $\Theta(n, m_n, r, \theta)$.

The cross-spectral matrix of the array output is given by $\boldsymbol{A}(r, \theta) = \boldsymbol{a}(r, \theta)\boldsymbol{a}(r, \theta)^{\mathrm{H}}$. The $N-1$ eigenvectors associated with the $N-1$ zero eigenvalues of $\boldsymbol{A}(r, \theta)$ span the noise subspace $\boldsymbol{G}(r, \theta)$. When $\boldsymbol{a}(\hat{r}, \hat{\theta}) = \boldsymbol{a}(r, \theta)$

$$\boldsymbol{a}(\hat{r}, \hat{\theta})^{\mathrm{H}}\boldsymbol{G}(r, \theta)\boldsymbol{G}(r, \theta)^{\mathrm{H}}\boldsymbol{a}(\hat{r}, \hat{\theta}) = 0 \tag{5}$$

where $\hat{r}$, $\hat{\theta}$ denotes estimates of the range and angle of the target, respectively. We can rewrite (5) as

$$\begin{aligned} &\boldsymbol{a}(\hat{r}, \hat{\theta})^{\mathrm{H}}\boldsymbol{G}(r, \theta)\boldsymbol{G}(r, \theta)^{\mathrm{H}}\boldsymbol{a}(\hat{r}, \hat{\theta}) \\ &= 2(N-1) - 2\cos[(\hat{\Theta}_1 - \hat{\Theta}_0) - (\Theta_1 - \Theta_0)] \\ &\quad -2\cos[(\hat{\Theta}_2 - \hat{\Theta}_0) - (\Theta_2 - \Theta_0)] - \cdots \\ &\quad -2\cos[(\hat{\Theta}_{N-1} - \hat{\Theta}_0) - (\Theta_{N-1} - \Theta_0)] = 0 \end{aligned} \tag{6}$$

Equation (6) specifies a linear system, with the kth equation

$$(\hat{\Theta}_k - \hat{\Theta}_0) - (\Theta_k - \Theta_0) = 2\pi l_k, \quad k = 1, 2, \ldots, N-1 \tag{7}$$

where $l_1, l_2, \ldots, l_{N-1}$ are arbitrary integers. Substituting $\Theta_n = 2\pi\left(n((2f_0 d\sin\theta)/c) - m_n(2\Delta f r/c)\right)$ into (7), the equations can be written as

$$\begin{bmatrix} 1 & m_0 - m_1 \\ 2 & m_0 - m_2 \\ \cdots & \cdots \\ N-1 & m_0 - m_{N-1} \end{bmatrix} \begin{bmatrix} \dfrac{2f_0 d}{c}(\sin\hat{\theta} - \sin\theta) \\ \dfrac{2\Delta f}{c}(\hat{r} - r) \end{bmatrix} = \begin{bmatrix} l_1 \\ l_2 \\ \cdots \\ l_{N-1} \end{bmatrix} \tag{8}$$

When the linear system has the unique solution $\mathbf{0}$, the range and angle can be uniquely identified with $\hat{r} = r$ and $\hat{\theta} = \theta$, respectively. Otherwise, we have ambiguous estimates. The number of solutions to (8) depends on the rank of the corresponding augmented matrix which depends on the array pattern m_n. To analyse the solutions to (8), we employ the following augmented matrix of the equations:

$$\begin{bmatrix} 1 & m_0 - m_1 & l_1 \\ 0 & (m_0 - m_2) - 2(m_0 - m_1) & l_2 - 2l_1 \\ \cdots & \cdots & \\ 0 & (m_0 - m_{N-1}) - (N-1)(m_0 - m_1) & l_{N-1} - (N-1)l_1 \end{bmatrix} \tag{9}$$

Consider the $N-2$ elements of the second column (other than the first element): $(m_0 - m_2) - 2(m_0 - m_1), \ldots, (m_0 - m_{N-1}) - (N-1)(m_0 - m_1)$. The greatest common divisor of these elements is

$$\begin{aligned} P = \gcd(&(m_0 - m_2) - 2(m_0 - m_1), \ldots, \\ &(m_0 - m_{N-1}) - (N-1)(m_0 - m_1)) \end{aligned} \tag{10}$$

Let Q denote the number of solutions to (8)

$$\text{if } P = 0 \quad Q = \infty \tag{11}$$

$$\text{if } P = 1 \quad Q = 1 \tag{12}$$

$$\text{if } P > 1 \quad Q = P \tag{13}$$

Equations (11)–(13) compose the proposed criterion for ambiguity identification. In particular, (12) signifies the unambiguous condition, which means (8) has the unique solution **0**, leading to unambiguous estimates of the range and angle. Both (11) and (13) signify the ambiguous conditions, under which (8) has other solutions besides **0**, which causes ambiguous range–angle estimates. The number of ambiguous localisations of (11) is infinite while the number of ambiguous localisations of (13) is $P - 1$.

It is worth noting that as the number of elements is larger, the possibility of an ambiguous frequency pattern becomes smaller.

Numerical results: Consider a uniform linear array. The reference carrier frequency is $f_0 = 3\,\text{GHz}$, whereas the frequency increment is $\Delta f = 1\,\text{MHz}$. The element spacing is $d = 0.025\,\text{m}$. We suppose one target located in the angle $\theta = 0^\circ$ and the slant range $r = 10\,\text{km}$.

First, we suppose the number of elements is $N = 4$. The number of all possible permutation frequency patterns is $N! = 24$. Using the proposed criterion, it can be shown that there exist 12 unambiguous frequency patterns and 12 ambiguous ones. We consider three frequency patterns, [0 1 2 3], [3 1 0 2], and [2 0 3 1], as examples. The greatest common divisor associated with these three frequency patterns computed by (10) is $P = 0$, $P = 1$, and $P = 5$, respectively. Fig. 1 illustrates the locus of the solution set of (8) and the multiple signal classification (MUSIC) spectrum using these three frequency patterns. In particular, Figs. 1*a*, *c,* and *e* show the loci of (8) with frequency patterns [0 1 2 3], [3 1 0 2], and [2 0 3 1], respectively, whereas Figs. 1*b*, *d,* and *f* are the corresponding MUSIC spectra. Note that solutions that meet all equations in (8) are the intersections of the solution locus of each individual equation. It is seen that the number of solutions to (8) is $Q = \infty$ with the frequency pattern [0 1 2 3], $Q = 1$ with [3 1 0 2], and $Q = 5$ with [2 0 3 1]. This verifies our analytical results expressed in (11)–(13). Moreover, the number of peaks of the MUSIC spectrum as shown in Figs. 1*b*, *d* and *f* is infinite, one, and greater than one (but finite), respectively, which again verifies our results (11)–(13). Therefore, the proposed criterion is effective in ambiguity identification for target localisation.

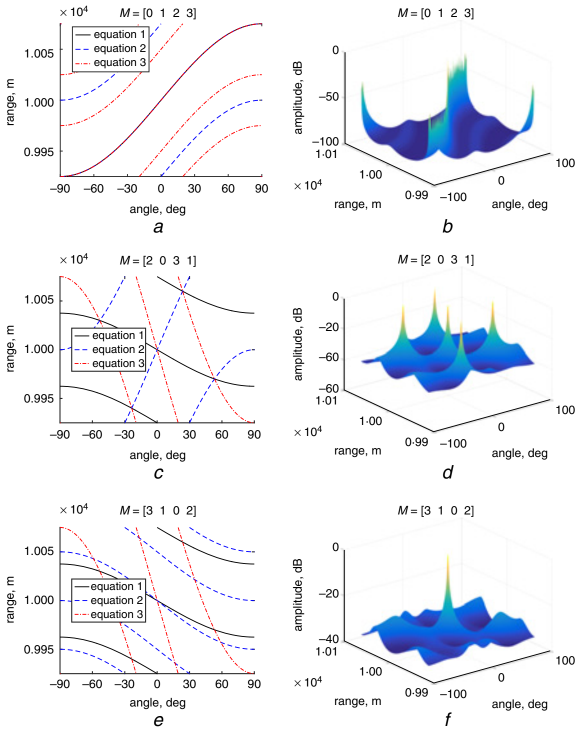

Fig. 1 *Example frequency patterns when* $N = 4$

a, *c*, *e* Solution locus of (8) with frequency patterns [0 1 2 3], [3 1 0 2], and [2 0 3 1]

b, *d*, *f* Corresponding MUSIC spectrum

Next, we examine how the likelihood of obtaining an ambiguous frequency pattern is affected by the number of array elements. Fig. 2 shows the percentage of ambiguous frequency patterns when the number of elements varies from $N = 4$ to 11. It is seen that the percentage of ambiguous frequency pattern reduces as the number of elements increase, and decrease below 1% when $N > 10$. The proposed criterion is more useful for small arrays.

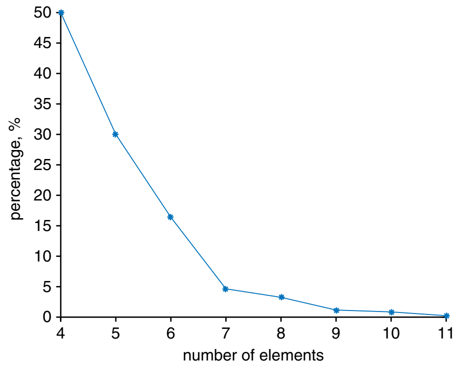

Fig. 2 *Percentage of ambiguous frequency pattern*

Conclusion: A simple criterion to identify unambiguous frequency patterns for target localisation was proposed. The criterion was derived by an eigenvector-based method. Simulations verify the effectiveness of the criterion.

Acknowledgments: This work was completed during the visit of the first author to Stevens Institute of Technology, with support by the Innovation Project of Guangxi Graduate Education.

Submitted: *19 June 2017* E-first: *18 August 2017*
doi: 10.1049/el.2017.2355
One or more of the Figures in this Letter are available in colour online.

Jingjing Li and Shan Ouyang (*School of Information and Communications, Guilin University of Electronic Technology, Guilin 541004, Guangxi, People's Republic of China*)

✉ E-mail: 1402101004@mails.guet.edu.cn

Hongbin Li (*Department of Electrical and Computer Engineering, Stevens Institute of Technology, Hoboken, NJ 07030, USA*)

References

1 Xu, J., Liao, G., Zhu, S., *et al.*: 'Deceptive jamming suppression with frequency diverse MIMO radar', *Signal Process.*, 2015, **113**, pp. 9–17
2 Xu, J., Liao, G., Zhu, S., *et al.*: 'Joint range and angle estimation using MIMO radar with frequency diverse array', *Trans. Signal Process.*, 2015, **63**, (13), pp. 3396–3410
3 Ran, D., and Jia, X.: 'Enhanced back projection algorithm for linear frequency diverse array synthetic aperture radar imaging'. Proc. VDE EUSAR'16, Hamburg, Germany, June 2016, pp. 1–5
4 Baizert, P., Hale, T.B., Temple, M.A., *et al.*: 'Forward-looking radar GMTI benefits using a linear frequency diverse array', *Electron. Lett.*, 2006, **42**, (22), pp. 1311–1312
5 Liu, Y., Ruan, H., Wang, L., *et al.*: 'The random frequency diverse array: a new antenna structure for uncoupled direction-range indication in active sensing', *J. Sel. Top. Signal Process.*, 2017, **11**, (2), pp. 295–308
6 Khan, W., Qureshi, I.M., and Saeed, S.: 'Frequency diverse array radar with logarithmically increasing frequency offset', *Antennas Wirel. Propag. Lett.*, 2015, **14**, pp. 499–502
7 Gao, K., Wang, W.Q., Cai, J., *et al.*: 'Decoupled frequency diverse array range-angle-dependent beampattern synthesis using non-linearly increasing frequency offsets', *Microw. Antennas Propag.*, 2016, **10**, (8), pp. 880–884
8 Sammartino, P.F., Baker, C.J., and Griffiths, H.D.: 'Range–angle dependent waveform'. Proc. IEEE Radar Conf., Arlington, VA, USA, May 2010, pp. 511–515
9 Jones, A.M.: 'Frequency diverse array receiver architectures', Master's thesis, Wright State University, 2011, pp. 16–17