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Computationally efficient parameter estimation for harmonic sinusoidal signals

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Abstract

A Markov-like weighted least squares (WLS) estimator is presented herein for harmonic sinusoidal parameter estimation. The estimator involves two distinct steps whereby it first obtains a set of initial parameter estimates that neglect the harmonic structure by some standard sinusoidal parameter estimation technique, and then the initial parameter estimates are refined via a WLS fit. Numerical results suggest that the proposed estimator achieves similar performance to the optimal nonlinear least-squares method for a moderate or large number of data samples and/or signal-to-noise ratio (SNR), but at a significantly reduced computational complexity. Furthermore, the former is observed to have a lower threshold SNR than the latter. © 2000 Elsevier Science B.V. All rights reserved.

Zusammenfassung

Es wird ein mit dem Markov-Schätzer verwandtes gewichtetes Kleinste-Quadrate Verfahren zur Schätzung der Parameter von harmonischen sinusförmigen Signalen vorgestellt. Das Verfahren besteht aus zwei Schritten. Zunächst werden Anfangsschätzungen mit Hilfe von Standardschätzern zur Bestimmung der Parameter von Sinussignalen gewonnen, wobei die harmonische Signalstruktur außer Acht gelassen wird. Anschließend werden die Anfangsschätzungen durch eine gewichtete Kleinste-Quadrate Anpasung verbessert. Numerische Untersuchungen verdeutlichen, daß der vorgeschlagene Schätzer bei erheblich reduziertem Rechenaufwand ähnlich gute Ergebnisse erzielt wie das optimale nichtlineare Kleinste-Quadrate Verfahren wenn die Zahl der Abtastwerte und/oder das SNR moderat bis grosß ist. Außerdem wird beobachtet, daß der vorgeschlagene Schätzer ein niedrigeres Schwellen-SNR bestizt als letzerer. © 2000 Elsevier Science B.V. All rights reserved.

Résumé

L'estimateur des moindres carrés pondérés de type Markov (WLS) est présenté ici pout l'estimation de paramètres sinusoïdaux harmoniques. L'estimateur implique deux étapes distinctes par lesquelles il obtient tout d'abord un ensemble

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d'estimées des paramètres initiaux qui négligent la structure harmonique par une technique d'estimation de paramètres sinusoïdaux standard, et ensuite les estimées des paramètres initiaux sont raffinés via un ajustement WLS. Les résultats numériques suggèrent que 1'estimateur proposé atteint des performances similaires à celles de la méthode des moindres carrés non linéaires optimale pour un nombre d'échantillons de données et/ou un rapport signal sur bruit (SNR) modérés ou larges, mais pour une complexité de calcul significativement réduite. De plus, on observe que le premier a un seuil de SNR plus bas que le dernier.

Keywords: Harmonic sinusoidal signal; Fundamental frequency estimation; Weighted least squares

1. Introduction

Consider the noise-corrupted measurements of a harmonic signal

$$y(n) = \sum_{k=1}^{K} \alpha_k e^{j(kn\omega_0 + \phi_k)} + v(n), \quad n = 0,1,\dots,N-1, \tag{1}$$

where ω_0 is the *fundamental frequency* with $0 < |\omega_0| < \pi/K$, $\alpha_k > 0$ and $\phi_k \in [0,2\pi)$ denote the amplitude and phase, respectively, of the kth harmonic component, and $v(n)$ is the complex white Gaussian noise with zero-mean and variance σ^2. Let $\boldsymbol{y} = [y(0) \dots y(N-1)]^{\mathrm{T}}$, where $(\cdot)^{\mathrm{T}}$ denotes the transpose, and $\boldsymbol{v} \in \mathbb{C}^{N\times 1}$ be similarly formed from $\{v(n)\}_{n=0}^{N-1}$. Then, (1) can be compactly rewritten as

$$\boldsymbol{y} = \boldsymbol{A}(\omega_0)\boldsymbol{\beta} + \boldsymbol{v}, \tag{2}$$

where $\boldsymbol{A}(\omega_0) \in \mathbb{C}^{N\times K}$ denotes the Vandermonde matrix with the kth column given by $\boldsymbol{a}_k = [1, \mathrm{e}^{\mathrm{j}k\omega_0}, \dots, \mathrm{e}^{\mathrm{j}k(N-1)\omega_0}]^{\mathrm{T}}$, and $\boldsymbol{\beta} = [\alpha_1, \mathrm{e}^{\mathrm{j}\phi_1}, \dots, \alpha_K \mathrm{e}^{\mathrm{j}\phi_K}]^{\mathrm{T}}$. It is assumed that the number of harmonic components, K, is known. (The estimation of K was discussed in, e.g., [12,13,15].) The problem of interest is to estimate ω_0 and $\{\alpha_k, \phi_k\}_{k=1}^{K}$ from the observations $\{y(n)\}_{n=0}^{N-1}$.

The above problem occurs in speech and musical signal processing, angular speed determination of rotating targets illuminated by a radar, passive detection and location of helicopters and boats, among others. While a rich literature exists dealing with the standard (i.e., non-harmonic) sinusoidal parameter estimation problem, starting from the early work of Whittle [16], to Hannan [1,2], and more recently by Schmidt and others (see [3,5–11] and references therein), the study on harmonic sinusoidal parameter estimation is relatively less emphasized. Specifically, Nehorai and Porat introduced an adaptive comb filtering technique for harmonic signal enhancement [4]. The method typically requires a relatively large number of observations to ensure its convergence. Quinn and Thomson considered a generalized least-squares approach to harmonic parameter estimation [14], which uses periodogram averaging to estimate the spectrum of the observation noise and, therefore, can be applied to colored noise applications. The generalized least-squares criterion in [14] is a nonlinear function of ω_0. Hence, a one-dimensional (1D) search is required, similar to the nonlinear least-squares (NLS) algorithm to be discussed next. Zeytinoğlu and Wong investigated the detection of harmonic signals [17]. It was shown therein that the knowledge of the harmonic structure and the fundamental frequency can be incorporated to significantly improve the detection performance.

Perhaps the reason why the attention to harmonic parameter estimation has been somewhat deficient may be that the NLS estimate of ω_0, which coincides with the optimal maximum likelihood (ML) estimate when $v(n)$ is white Gaussian, can be obtained by using a seemingly simple 1D search of the NLS cost function. Specifically, the NLS estimates of the unknown parameters are determined by (see, e.g., [7])

$$\hat{\omega}_0 = \arg\min_{\omega_0} \boldsymbol{y}^{\mathrm{H}}\{\boldsymbol{I}_N - \boldsymbol{A}(\omega_0)[\boldsymbol{A}^{\mathrm{H}}(\omega_0)\boldsymbol{A}(\omega_0)]^{-1} \times \boldsymbol{A}^{\mathrm{H}}(\omega_0)\}\boldsymbol{y}, \tag{3}$$

$$\hat{\boldsymbol{\beta}} = [\boldsymbol{A}^{\mathrm{H}}(\omega_0)\boldsymbol{A}(\omega_0)]^{-1}\boldsymbol{A}^{\mathrm{H}}(\omega_0)\boldsymbol{y}|_{\omega_0=\hat{\omega}_0}, \tag{4}$$

where $\boldsymbol{I}_N$ is the $N \times N$ identity matrix. Unfortunately, the NLS cost function in (3) is usually multimodal with many local minima. An example of the NLS cost function is shown in Fig. 1 where $K = 5$, $N = 32$, $\sigma^2 = 0.1$, $\omega_0 = 2\pi \times 0.08$, $\alpha_k = 1$ and $\phi_k = \pi/4$, $\forall k$. Hence, the minimization of the NLS cost function requires the use of a very fine searching algorithm and may be computationally prohibitive.

In this paper, we present a new method for the parameter estimation of harmonic sinusoidal signals. The proposed method makes use of some initial parameter estimates obtained by, for example, the MUSIC algorithm [10] which ignores the harmonic structure. Then, the estimates are refined by using a Markov-like weighted least squares (WLS) technique. We show using numerical examples that the method is computationally much more efficient than the NLS method, and yet it performs similarly to the latter and is very close to the Cramér–Rao bound (CRB), the best performance bound of any unbiased estimators, for moderate or large N and/or signal-to-noise ratio (SNR).

2. The Markov-like WLS estimator

The Markov-like WLS estimator first ignores the harmonic structure and uses some standard sinusoidal parameter estimator, such as MUSIC [10], to obtain the initial estimates, $\tilde{\omega}_k$ of $\omega_k = k\omega_0$, for $k = 1, \dots, K$. The initial estimates, $\{\tilde{\alpha}_k, \tilde{\phi}_k\}_{k=1}^K$ of $\{\alpha_k, \phi_k\}_{k=1}^K$, can be obtained via least squares [from (4) with $k\omega_0$ replaced by $\tilde{\omega}_k$]. Next, let

$$\boldsymbol{\theta}' = [\alpha_1, \phi_1, \omega_0, \dots, \alpha_K, \phi_K, K\omega_0]^{\mathrm{T}} \in \mathbb{R}^{3K \times 1},$$

$$\boldsymbol{\eta}' = [\omega_0, \alpha_1, \phi_1, \dots, \alpha_K, \phi_K]^{\mathrm{T}} \in \mathbb{R}^{(2K+1) \times 1}.$$

Apparently, there exists a matrix $\boldsymbol{S}' \in \mathbb{Z}^{3K \times (2K+1)}$ that has full column rank so that

$$\boldsymbol{\theta}' = \boldsymbol{S}'\boldsymbol{\eta}'. \tag{5}$$

The MUSIC estimate $\tilde{\boldsymbol{\theta}}'$ of $\boldsymbol{\theta}'$ is not statistically efficient. However, for sufficiently large N and M, where M is length of the data subvectors used in MUSIC [10], its performance is close to the CRB corresponding to $\boldsymbol{\theta}'$ [6]. Hence, a Markov-like

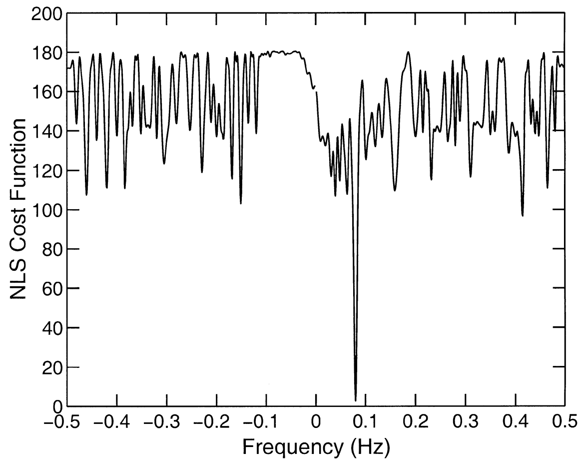

Fig. 1. An example of the NLS cost function versus frequency.

WLS estimate $\hat{\boldsymbol{\eta}}'$ of $\boldsymbol{\eta}'$ can be obtained as

$$\hat{\boldsymbol{\eta}}' = \arg\min_{\boldsymbol{\eta}'} \|\tilde{\boldsymbol{\theta}}' - \boldsymbol{S}'\boldsymbol{\eta}'\|^2_{\boldsymbol{W}'}, \tag{6}$$

where the weighting matrix $\boldsymbol{W}' = \mathrm{CRB}^{-1}(\tilde{\boldsymbol{\theta}}')$, with $\mathrm{CRB}(\tilde{\boldsymbol{\theta}}') \in \mathbb{R}^{3K \times 3K}$ being the CRB matrix that does not assume the knowledge of the harmonic structure and is evaluated at $\tilde{\boldsymbol{\theta}}'$. For large N, $\boldsymbol{W}'$ can be well approximated by [9]

$$\boldsymbol{W}' \simeq \begin{bmatrix} \boldsymbol{W}'_1 & & 0 \\ & \ddots & \\ 0 & & \boldsymbol{W}'_K \end{bmatrix}, \tag{7}$$

where

$$\boldsymbol{W}'_k = \frac{1}{\sigma^2} \begin{bmatrix} N & 0 & 0 \\ 0 & 2N\tilde{\alpha}_k^2 & N^2\tilde{\alpha}_k^2 \\ 0 & N^2\tilde{\alpha}_k^2 & \frac{2}{3}N^3\tilde{\alpha}_k^2 \end{bmatrix}, \quad k = 1,2,\dots,K. \tag{8}$$

The fact that $\boldsymbol{W}'_k$ is block diagonal implies that the amplitude estimates in $\hat{\boldsymbol{\eta}}'$ and $\tilde{\boldsymbol{\theta}}'$ are equivalent, i.e., applying WLS does not lead to refined estimates of $\{\alpha_k\}_{k=1}^K$. Hence, the cost function in (6) can be simplified by excluding the amplitude parameters.

Let $\boldsymbol{\theta} \in \mathbb{R}^{2K \times 1}$ and $\boldsymbol{\eta} \in \mathbb{R}^{(K+1) \times 1}$ be similarly defined to $\boldsymbol{\theta}'$ and $\boldsymbol{\eta}'$, respectively, except that the amplitude parameters $\{\alpha_k\}_{k=1}^K$ have been removed. Then there exists a matrix $\boldsymbol{S} \in \mathbb{Z}^{2K \times (K+1)}$ with full column rank so that

$$\boldsymbol{\theta} = \boldsymbol{S}\boldsymbol{\eta}. \tag{9}$$

Let $\tilde{\boldsymbol{\theta}}$ be the corresponding initial estimate of $\boldsymbol{\theta}$. Similarly to (6), the Markov-like WLS estimate $\hat{\boldsymbol{\eta}}$ of $\boldsymbol{\eta}$ is given by

$$\hat{\boldsymbol{\eta}} = \arg\min_{\boldsymbol{\eta}} \|\tilde{\boldsymbol{\theta}} - \boldsymbol{S}\boldsymbol{\eta}\|^2_{\boldsymbol{W}}, \tag{10}$$

where $\boldsymbol{W} \in \mathbb{R}^{2K\times 2K}$ is block diagonal with the diagonal blocks obtained from (8) by deleting the first row and column of each $\boldsymbol{W}'_k$.

The solution to (10) is easily obtained by exploiting the fact that $\boldsymbol{W}$ is block-diagonal. Specifically, the cost function in (10) can be rewritten as

$$\begin{aligned} J(\boldsymbol{\eta}) &\triangleq \|\tilde{\boldsymbol{\theta}} - \boldsymbol{S}\boldsymbol{\eta}\|^2_{\boldsymbol{W}} \\ &= \frac{1}{\sigma^2}\sum_{k=1}^{K} \left\{\begin{bmatrix}\tilde{\phi}_k \\ \tilde{\omega}_k\end{bmatrix} - \begin{bmatrix}\phi_k \\ k\omega_0\end{bmatrix}\right\}^{\mathrm{T}} \\ &\quad \times \begin{bmatrix} 2N\tilde{\alpha}_k^2 & N^2\tilde{\alpha}_k^2 \\ N^2\tilde{\alpha}_k^2 & \frac{2}{3}N^3\tilde{\alpha}_k^2 \end{bmatrix} \left\{\begin{bmatrix}\tilde{\phi}_k \\ \tilde{\omega}_k\end{bmatrix} - \begin{bmatrix}\phi_k \\ k\omega_0\end{bmatrix}\right\} \\ &= \frac{1}{\sigma^2}\sum_{k=1}^{K} \tilde{\alpha}_k^2 \Big\{[2N(\tilde{\phi}_k - \phi_k) \\ &\quad + N^2(\tilde{\omega}_k - k\omega_0)](\tilde{\phi}_k - \phi_k) \\ &\quad + \left[N^2(\tilde{\phi}_k - \phi_k) + \frac{2}{3}N^3(\tilde{\omega}_k - k\omega_0)\right] \\ &\quad \times (\tilde{\omega}_k - k\omega_0)\Big\}. \end{aligned} \tag{11}$$

Differentiating the above equation with respect to (w.r.t.) ϕ_k and equating the result to zero, we obtain

$$\frac{\partial J}{\partial \phi_k} = 0 = \frac{\tilde{\alpha}_k^2}{\sigma^2}[-4N(\tilde{\phi}_k - \phi_k) - 2N^2(\tilde{\omega}_k - k\omega_0)]. \tag{12}$$

It follows that, given ω_0, the estimate of ϕ_k is

$$\hat{\phi}_k = \tilde{\phi}_k + \frac{N}{2}(\tilde{\omega}_k - k\omega_0), \quad k = 1,2,\dots,K. \tag{13}$$

Substituting the above equation in (11) and observing that $2N(\tilde{\phi}_k - \phi_k) + N^2(\tilde{\omega}_k - k\omega_0) = 0$, we have

$$\begin{aligned} J(\omega_0) &= \frac{1}{\sigma^2}\sum_{k=1}^{K} \tilde{\alpha}_k^2[N^2(\tilde{\phi}_k - \phi_k) + \tfrac{2}{3}N^3(\tilde{\omega}_k - k\omega_0)] \\ &\quad \times (\tilde{\omega}_k - k\omega_0) \\ &= \frac{N^3}{6\sigma^2}\sum_{k=1}^{K} \tilde{\alpha}_k^2(\tilde{\omega}_k - k\omega_0)^2. \end{aligned} \tag{14}$$

Differentiating the above cost function w.r.t. ω_0 yields the WLS estimate of ω_0:

$$\hat{\omega}_0 = \frac{\sum_{k=1}^{K} k\tilde{\alpha}_k^2\tilde{\omega}_k}{\sum_{k=1}^{K} k^2\tilde{\alpha}_k^2}. \tag{15}$$

Using the above $\hat{\omega}_0$ to replace ω_0 in (13), we obtain the WLS estimates of ϕ_k. Hence (13) and (15) provide closed-form expressions for the WLS estimates of the phase and frequency parameters. Observe that $\hat{\omega}_0$ in (15) is a weighted linear regression over $\{\tilde{\omega}_k\}_{k=1}^K$ and does not depend on $\tilde{\phi}_k$. If all α_k are known a priori to be equal, then $\hat{\omega}_0$ is a simple linear regression (average) over $\{\tilde{\omega}_k\}_{k=1}^K$.

It should be noted that a new set of phase estimates, which are different from those in (13) (with ω_0 replaced by $\hat{\omega}_0$), can be obtained from (4) by using $\hat{\omega}_0$ in (15). We have observed that the so-obtained phase estimates are generally more accurate and should be preferred, especially when the SNR is relatively low.

3. Cramér–Rao bound

An asymptotic (for large N) CRB for the case of real-valued harmonic sinusoidal signals was derived in [4]. In the following, we derive the *exact* CRB for the parameter estimation problem posed for the complex data model in (1).

Let $\boldsymbol{\psi} = [\omega_0,\alpha_1,\dots,\alpha_K,\phi_1,\dots,\phi_K]^{\mathrm{T}} \triangleq [\omega_0,\boldsymbol{\alpha}^{\mathrm{T}},\boldsymbol{\phi}^{\mathrm{T}}]^{\mathrm{T}} \in \mathbb{R}^{(2K+1)\times 1}$. Observe that $\boldsymbol{\psi}$ is a permutation of $\boldsymbol{\eta}'$. We rewrite (2) as

$$\boldsymbol{y} = \boldsymbol{B}(\omega_0,\boldsymbol{\phi})\boldsymbol{\alpha} + \boldsymbol{v} \triangleq \boldsymbol{x}(\boldsymbol{\psi}) + \boldsymbol{v}, \tag{16}$$

where $\boldsymbol{B}(\omega_0,\boldsymbol{\phi}) \in \mathbb{C}^{N\times K}$ with its kth column defined by $\boldsymbol{b}_k = \mathrm{e}^{\mathrm{j}\phi_k}\boldsymbol{a}_k$. By using the Slepian–Bangs formula (see, e.g., [1]), the CRB matrix for the problem under study is given by

$$\mathrm{CRB}^{-1}(\boldsymbol{\psi}) = \frac{2}{\sigma^2}\mathrm{Re}\left[\frac{\partial \boldsymbol{x}^{\mathrm{H}}(\boldsymbol{\psi})}{\partial \boldsymbol{\psi}}\frac{\partial \boldsymbol{x}(\boldsymbol{\psi})}{\partial \boldsymbol{\psi}^{\mathrm{T}}}\right], \tag{17}$$

where $(\cdot)^{\mathrm{H}}$ denotes the Hermitian transpose. Next, we evaluate the partial derivatives in (17) as follows:

$$\frac{\partial \boldsymbol{x}(\boldsymbol{\psi})}{\partial \boldsymbol{\psi}^{\mathrm{T}}} = \left[\frac{\partial \boldsymbol{x}(\boldsymbol{\psi})}{\partial \omega_0}, \frac{\partial \boldsymbol{x}(\boldsymbol{\psi})}{\partial \boldsymbol{\alpha}^{\mathrm{T}}}, \frac{\partial \boldsymbol{x}(\boldsymbol{\psi})}{\partial \boldsymbol{\phi}^{\mathrm{T}}}\right], \tag{18}$$

where

$$\frac{\partial \boldsymbol{x}(\boldsymbol{\psi})}{\partial \omega_0} = \boldsymbol{D}\alpha, \tag{19}$$

$$\frac{\partial \boldsymbol{x}(\boldsymbol{\psi})}{\partial \boldsymbol{\alpha}^{\mathrm{T}}} = \boldsymbol{B}(\omega_0,\boldsymbol{\phi}), \tag{20}$$

$$\frac{\partial \boldsymbol{x}(\boldsymbol{\psi})}{\partial \boldsymbol{\phi}^{\mathrm{T}}} = \mathrm{j}\boldsymbol{B}(\omega_0,\boldsymbol{\phi})\mathrm{diag}\{\alpha_1,\dots,\alpha_K\} \triangleq \mathrm{j}\boldsymbol{B}(\omega_0,\boldsymbol{\phi})\boldsymbol{C}, \tag{21}$$

with the kth column of $\boldsymbol{D} \in \mathbb{C}^{N\times K}$ given by

$$\boldsymbol{d}_k = \mathrm{j}k\mathrm{e}^{\mathrm{j}\phi_k}[0,\ \mathrm{e}^{\mathrm{j}k\omega_0},\dots,(N-1)\mathrm{e}^{\mathrm{j}k(N-1)\omega_0}]^{\mathrm{T}}. \tag{22}$$

It follows that

$$\mathrm{CRB}(\boldsymbol{\psi}) = \frac{\sigma^2}{2}\left\{\mathrm{Re}\begin{bmatrix} \boldsymbol{\alpha}^{\mathrm{T}}\boldsymbol{D}^{\mathrm{H}}\boldsymbol{D}\boldsymbol{\alpha} & \boldsymbol{\alpha}^{\mathrm{T}}\boldsymbol{D}^{\mathrm{H}}\boldsymbol{B} & \mathrm{j}\boldsymbol{\alpha}^{\mathrm{T}}\boldsymbol{D}^{\mathrm{H}}\boldsymbol{B}\boldsymbol{C} \\ \boldsymbol{B}^{\mathrm{H}}\boldsymbol{D}\alpha & \boldsymbol{B}^{\mathrm{H}}\boldsymbol{B} & \mathrm{j}\boldsymbol{B}^{\mathrm{H}}\boldsymbol{B}\boldsymbol{C} \\ -\mathrm{j}\boldsymbol{C}^{\mathrm{T}}\boldsymbol{B}^{\mathrm{H}}\boldsymbol{D}\boldsymbol{\alpha} & -\mathrm{j}\boldsymbol{C}^{\mathrm{T}}\boldsymbol{B}^{\mathrm{H}}\boldsymbol{B} & \boldsymbol{C}^{\mathrm{T}}\boldsymbol{B}^{\mathrm{H}}\boldsymbol{B}\boldsymbol{C} \end{bmatrix}\right\}^{-1}, \tag{23}$$

where we have omitted the dependence of $\boldsymbol{B}$ on ω_0 and $\boldsymbol{\phi}$ for notational simplicity.

4. Numerical results

We compare the performance of our proposed estimator and the optimal NLS method, which are referred to as the WLS and NLS estimators, respectively. We also compare the performance of the estimators with the CRB. For both estimators, we use MUSIC to obtain the initial parameter estimates. Specifically, we take the smallest frequency estimate (and ignore the others) obtained by MUSIC as the initial estimate of ω_0. The NLS cost function is minimized by using a gradient-type nonlinear optimization routine, *fminu*, provided in MATLAB. The signal consists of $K = 5$ harmonic components corrupted by a zero-mean complex white Gaussian noise, with $\omega_0 = 2\pi \times 0.08$, $\alpha_k = 1$ and $\phi_k = \pi/4$, $\forall k$. The SNR for the kth harmonic component is defined as $10\log_{10}\alpha_k^2/\sigma^2$(dB). All results shown below are based on 200 independent trials.

The first example considers the effect of SNR on the parameter estimation accuracy. Fig. 2(a) shows the root mean squared errors (RMSEs) of the estimates of ω_0, along with the CRB, as a function of the SNR for the first harmonic component when

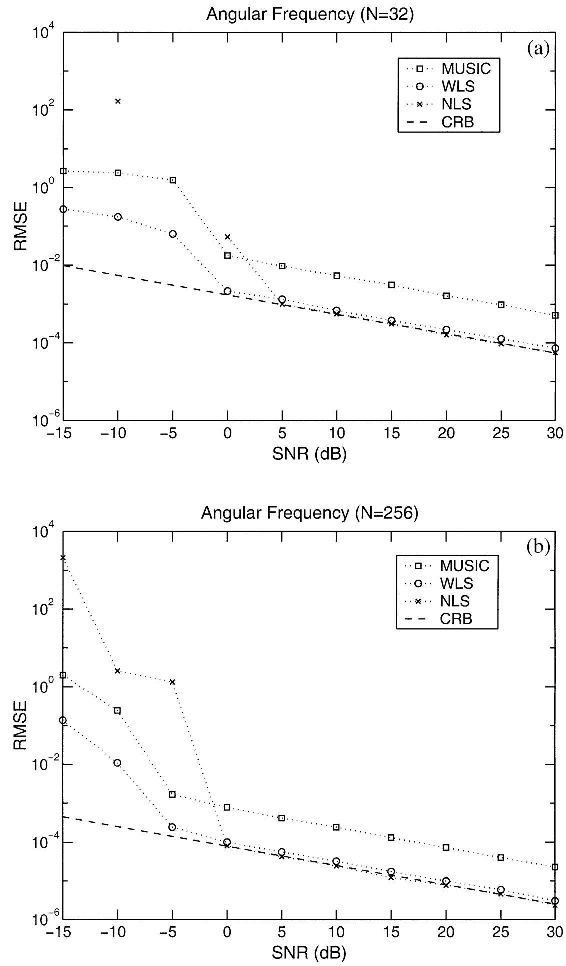

Fig. 2. RMSEs of the estimates of ω_0 and the associated CRB versus the SNR. (a) $N = 32$ and $M = 15$. (b) $N = 256$ and $M = 50$.

$N = 32$ and $M = 15$ (M is the length of the data subvectors used in MUSIC). The curves corresponding to the amplitude and phase estimation are similar to those for the frequency estimation and thus are not shown here. The performance of MUSIC is included in the plot to show the merit of incorporating the knowledge of the harmonic structure. It is seen that, for SNR $\geqslant 0$ dB, the performance of WLS is very close to the CRB. For small SNRs, the poor initial condition provided by MUSIC may cause NLS to converge to some local minima, as seen in the figure when SNR $= -10$ and 0 dB, or even not to converge at all, which is the case when SNR $= -15$ and -5 dB in this example. It is also noted that WLS has a lower threshold SNR than NLS. Fig. 2(b) shows the counterpart results when $N = 256$ and $M = 50$. It is observed that WLS and NLS behave similarly as in the previous figure except that the threshold SNR is decreased for both algorithms.

Next, we examine the effect of the data length, N, on the performance of the methods under study. The SNR for the first harmonic is fixed at some value and N is varied from 12 to 200. For each value of N, M is chosen as $N/2$ rounded up to the nearest integer. The other parameters are the same as in the previous example. Fig. 3(a) shows the RMSEs of the estimates of ω_0 as well as the CRB when SNR $= 5$ dB. We observe that even for relatively large N (such as when $N = 146$ in this example), the NLS algorithm may converge to some local minimum (due to poor initial estimates), whereas the WLS method is very close to the CRB for $N > 20$. Fig. 3(b) shows the results when the SNR is increased to 30 dB. In this case WLS and NLS are seen to perform similarly to one another for $N > 20$, with the latter being slightly better than the former. The performance of WLS can be slightly improved by using MODE [5] instead of MUSIC to provide the initial parameter estimates when the SNR is moderate or high. This is because MODE usually yields more accurate initial estimates than MUSIC in such a case.

Fig. 4 shows the numbers of MATLAB flops required by the WLS and NLS algorithms in the estimation of ω_0 in the previous example. The number of flops involved in the estimation of the amplitude and phase parameters is not counted

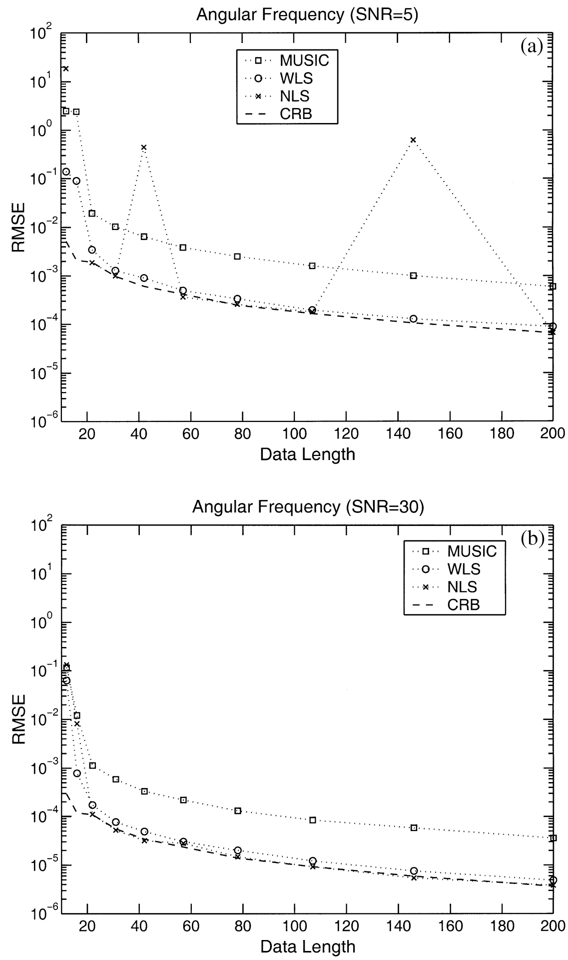

Fig. 3. RMSEs of the estimates of ω_0 and the associated CRB versus N. (a) SNR $= 5$ dB. (b) SNR $= 30$ dB.

since both WLS and NLS use (4) (and hence amount to the same number of flops) to obtain estimates for these parameters. Neither do we count the number of flops involved in the MUSIC algorithm for the same reason. Since NLS is iterative and the associated number of flops may vary from trial to trial, the numbers shown in Fig. 4 for NLS are the averages over 200 trials. It is seen from this figure that WLS is computationally much simpler than NLS. Also observe that whereas the complexity of WLS is not affected by N, NLS in general becomes more involved as N increases, which is explained by the fact the NLS cost function becomes more and more erratic as N increases

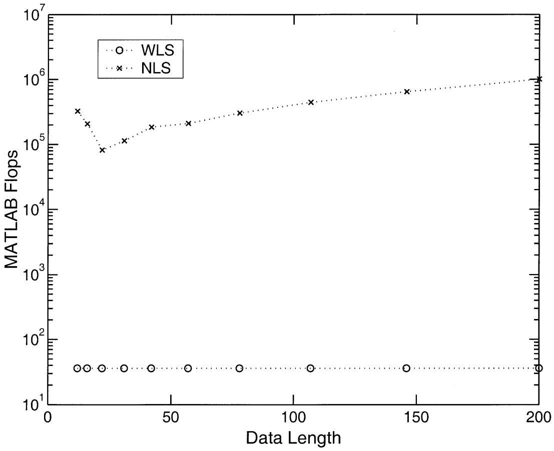

Fig. 4. MATLAB flops versus N.

[8]. We see that the number of flops involved in the NLS estimator may increase when N becomes too small. This is due to the poor initial estimates which makes the NLS estimator more difficult to converge.

5. Conclusions

We have presented a Markov-like WLS estimator for harmonic sinusoidal parameter estimation. We have shown through numerical examples that the proposed estimator performs similarly to the optimal NLS method when the SNR and/or N are moderate or high, but the former is computationally significantly simpler than the latter.

The proposed WLS estimator may also be extended to estimate the parameters of superimposed harmonic sinusoidal signals in which we have several harmonic signals present in the observed data. The extension would be straightforward if the harmonic signals are well-separated, i.e., the fundamental frequency of any harmonic signal is not close to any multiples of the other fundamental frequencies. Without this condition, it would be difficult to identify the fundamental frequencies associated with each harmonic signal from the initial frequency estimates. The reason for considering the WLS estimator for superimposed harmonic sinusoidal parameter estimation is that the NLS method in this case will need a search over an L-dimensional parameter space, where L is the number of harmonic signals, whereas the WLS estimator will still have a closed-form solution. As a result, the computational advantage of the WLS estimator over the NLS method will be even more striking. However, how to obtain a set of reliable initial estimates of the fundamental frequencies when the harmonic signals are not well-separated remains a future topic of research.

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