A New Derivation of the APES Filter

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***Abstract*— We introduce a novel design criterion for data-dependent narrowband filters that are of interest in temporal or spatial spectral analysis applications. The solution to the design problem considered is shown to coincide with the recently introduced amplitude and phase estimation (APES) filter. The new derivation of APES in this note sheds more light on the properties of APES and provides some intuitive explanation of the performance superiority of the APES filter over the Capon filter.**

***Index Terms*— Amplitude estimation, filter design, spectral analysis.**

I. Introduction

FILTERBANK approaches are commonly used for both temporal and spatial spectral analysis in either one or several dimensions. For any of these approaches, the key ingredient is a narrowband filter that is swept through the frequency interval of interest. Data-dependent (also called "adaptive") filters outperform data-independent filters and are hence preferred in most applications. A well-known data-dependent narrowband filter was derived by Levin *et al.* some 30 years ago (see the references in [1]); it is usually referred to as the Capon filter. More recently, Li and Stoica [2] derived a data-dependent filter with enhanced performance, which they called the amplitude and phase estimation (APES) filter. The APES filter was originally derived by means of a relatively involved approximate maximum likelihood (AML) approach [2]. Later it was also given a matched-filterbank interpretation [3]. The matched-filterbank derivation of APES is much simpler than the ML-based derivation. However, the former does not completely stand by itself because it uses an AML estimate of the residual covariance matrix obtained by the latter derivation. In this note, we obtain *the APES filter from pure narrowband-filter design considerations*. This provides a simple and self-contained derivation of this apparently fundamental filter, which comes out in so many ways and appears to outperform many other available filters.

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II. Forward-Only APES Filter

For the sake of conciseness, we focus on temporal one-dimensional data processing (yet, perfectly similar arguments apply to spatial data and to data in several dimensions). Let $\{y(n) \in \mathbb{C}\}_{n=0}^{N-1}$ denote the data sequence at hand, and let us assume that we want to design a finite impulse response (FIR) filter that passes the frequency ω in $\{y(n)\}$ without distortion and, at the same time, attenuates all the other frequencies "as much as possible." If we let $\boldsymbol{h} \in \mathbb{C}^{M\times 1}$ be the vector comprising the filter coefficients, then the filter output can be written as $\boldsymbol{h}^H\boldsymbol{y}(l)$, where $\boldsymbol{y}(l) = [y(l)\cdots y(l+M-1)]^T$ is the vector of forward-only data. Here, $(\cdot)^T$ and $(\cdot)^H$ denote the transpose and conjugate transpose, respectively.

The Capon filter is the solution to the following problem:

$$\min_{\boldsymbol{h}} \boldsymbol{h}^H\hat{\boldsymbol{R}}\boldsymbol{h}, \quad \text{subject to} \quad \boldsymbol{h}^H\boldsymbol{a}(\omega) = 1 \tag{1}$$

and is given by (assuming $\hat{\boldsymbol{R}}^{-1}$ exists)

$$\boldsymbol{h}_{\text{Capon}} = \frac{\hat{\boldsymbol{R}}^{-1}\boldsymbol{a}(\omega)}{\boldsymbol{a}^H(\omega)\hat{\boldsymbol{R}}^{-1}\boldsymbol{a}(\omega)} \tag{2}$$

where $\boldsymbol{a}(\omega) = [1 \quad e^{j\omega} \quad \cdots \quad e^{j(M-1)\omega}]^T$ and $\hat{\boldsymbol{R}} = (1/L)\ \Sigma_{l=0}^{L-1}\ \boldsymbol{y}(l)\boldsymbol{y}^H(l)$ with $L = N - M + 1$. Despite the intuitive appeal of the above design problem, the Capon filter was found to have a rather unsatisfactory behavior in some cases. More exactly, the filter compromises the "noise gain" (that is, $\boldsymbol{h}^H\boldsymbol{h}$) for the capability to cancel strong interference in the data, located at frequencies different from ω. The consequences are often poor estimates (biased downward) of the amplitude and phase of the component in the data at frequency ω [2], [3].

In this note, we consider another intuitively appealing (new) formulation of the design objective:

$$\min_{\boldsymbol{h},\alpha} \frac{1}{L}\sum_{l=0}^{L-1} |\boldsymbol{h}^H\boldsymbol{y}(l) - \alpha e^{j\omega l}|^2, \quad \text{subject to} \quad \boldsymbol{h}^H\boldsymbol{a}(\omega) = 1 \tag{3}$$

(once again, for any given ω). Because the filter output is now required by design to be as close as possible to a sinusoid with frequency ω, the noise gain of the resulting filter should be (much) more acceptable than that of the Capon filter. Let $\boldsymbol{g}(\omega) = (1/L)\ \Sigma_{l=0}^{L-1}\ \boldsymbol{y}(l)e^{-j\omega l}$. A straightforward calculation shows that the criterion function in (3) can be rewritten as

$$\begin{aligned}\frac{1}{L}\sum_{l=0}^{L-1} &|\boldsymbol{h}^H\boldsymbol{y}(l) - \alpha e^{j\omega l}|^2 \\ &= \boldsymbol{h}^H\hat{\boldsymbol{R}}\boldsymbol{h} - \alpha^*\boldsymbol{h}^H\boldsymbol{g}(\omega) - \alpha\boldsymbol{g}^H(\omega)\boldsymbol{h} + |\alpha|^2 \\ &= |\alpha - \boldsymbol{h}^H\boldsymbol{g}(\omega)|^2 + \boldsymbol{h}^H\hat{\boldsymbol{R}}\boldsymbol{h} - |\boldsymbol{h}^H\boldsymbol{g}(\omega)|^2\end{aligned} \tag{4}$$

where $(\cdot)^*$ denotes the complex conjugate. The minimization of (4) with respect to α is given by

$$\hat{\alpha} = \boldsymbol{h}^H \boldsymbol{g}(\omega). \tag{5}$$

Insertion of (5) in (4) yields the following minimization problem for the determination of $\boldsymbol{h}$:

$$\min_{\boldsymbol{h}} \boldsymbol{h}^H \hat{\boldsymbol{Q}}(\omega)\boldsymbol{h} \quad \text{subject to} \quad \boldsymbol{h}^H \boldsymbol{a}(\omega) = 1 \tag{6}$$

where $\hat{\boldsymbol{Q}}(\omega) = \hat{\boldsymbol{R}} - \boldsymbol{g}(\omega)\boldsymbol{g}^H(\omega)$. The solution to (6) is readily obtained [in effect, observe that (6) is of the same type as (1)]:

$$\boldsymbol{h}_{\text{FAPES}} = \frac{\hat{\boldsymbol{Q}}^{-1}(\omega)\boldsymbol{a}(\omega)}{\boldsymbol{a}^H(\omega)\hat{\boldsymbol{Q}}^{-1}(\omega)\boldsymbol{a}(\omega)}. \tag{7}$$

This is recognized as the forward-only APES (FAPES) filter, and the $\hat{\alpha}$ in (5) as the FAPES estimate of the "complex spectrum" [2], [3].

III. Forward-Backward APES Filter

Forward-backward (FB) averaging has been used for enhanced performance in many spectral analysis applications. For example, it was shown in [4] that the bias of the Capon complex spectral estimator can be halved by using this technique. In the previous section, we obtained the APES filter by using only forward data vectors. We show below that forward-backward averaging can be readily incorporated into the proposed filter design criterion. Let the backward data vectors be constructed as $\tilde{\boldsymbol{y}}(l) = [y^*(N-l-1) \ \cdots \ y^*(N-l-M)]^T$. We wish that the outputs obtained by running the data through the filter both forward and backward are as close as possible to a sinusoid with frequency ω. This design objective can be written as

$$\min_{\boldsymbol{h},\alpha,\beta} \frac{1}{2L} \sum_{l=0}^{L-1} \{|\boldsymbol{h}^H \boldsymbol{y}(l) - \alpha e^{j\omega l}|^2 + |\boldsymbol{h}^H \tilde{\boldsymbol{y}}(l) - \beta e^{j\omega l}|^2\},$$
$$\text{subject to} \quad \boldsymbol{h}^H \boldsymbol{a}(\omega) = 1. \tag{8}$$

The minimization of (8) with respect to α and β gives $\hat{\alpha} = \boldsymbol{h}^H \boldsymbol{g}(\omega)$ and $\hat{\beta} = \boldsymbol{h}^H \tilde{\boldsymbol{g}}(\omega)$, where $\tilde{\boldsymbol{g}}(\omega) = (1/L) \ \Sigma_{l=0}^{L-1} \ \tilde{\boldsymbol{y}}(l)e^{-j\omega l}$. It follows that (8) leads to

$$\min_{\boldsymbol{h}} \boldsymbol{h}^H \hat{\boldsymbol{Q}}_{\text{FB}}(\omega)\boldsymbol{h}, \quad \text{subject to} \quad \boldsymbol{h}^H \boldsymbol{a}(\omega) = 1 \tag{9}$$

where $\hat{\boldsymbol{Q}}_{\text{FB}}(\omega) = \frac{1}{2}(\hat{\boldsymbol{R}} + \tilde{\boldsymbol{R}}) - \frac{1}{2}[\boldsymbol{g}(\omega)\boldsymbol{g}^H(\omega) + \tilde{\boldsymbol{g}}(\omega)\tilde{\boldsymbol{g}}^H(\omega)]$ with $\tilde{\boldsymbol{R}} = (1/L) \ \Sigma_{l=0}^{L-1} \ \tilde{\boldsymbol{y}}(l)\tilde{\boldsymbol{y}}^H(l)$. The solution of (9) is given by [similar to (7)]

$$\boldsymbol{h}_{\text{FBAPES}} = \frac{\hat{\boldsymbol{Q}}_{\text{FB}}^{-1}(\omega)\boldsymbol{a}(\omega)}{\boldsymbol{a}^H(\omega)\hat{\boldsymbol{Q}}_{\text{FB}}^{-1}(\omega)\boldsymbol{a}(\omega)} \tag{10}$$

which is recognized as the forward-backward APES (FBAPES) filter [2], [4]. Note that we could have imposed $\beta = \alpha^* e^{-j(N-1)\omega}$ in (8). Though we made no such constraint while deriving the $\boldsymbol{h}_{\text{FBAPES}}$ above, it turns out that $\boldsymbol{h}_{\text{FBAPES}}$ satisfies the symmetry property [2], $\boldsymbol{J}\boldsymbol{h}^* = \boldsymbol{h}e^{-j(M-1)\omega}$, where $\boldsymbol{J}$ denotes the exchange matrix whose antidiagonal elements are ones and the remaining elements are zeros. Hence, one can easily show that $\hat{\beta} = \hat{\alpha}^* e^{-j(N-1)\omega}$.

IV. Concluding Remarks

The APES filter has been found to outperform the Capon filter in several important applications, such as line spectral analysis, continuous-spectrum estimation, radar target feature extraction, and synthetic aperture radar imagery [2]. At the same time, the computation of the APES filter is only marginally more demanding than that of the Capon filter [3], [5]. Additionally, we have shown here that the APES filter, similarly to the Capon filter, can be derived from simple and intuitively appealing filter design considerations. All of these facts suggest that APES may be a more preferable approach than Capon's for many applications.

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