Assignment 5

5. **(a)** $\int_{-1}^{1} \frac{c}{\sqrt{1-x^2}}\,dx = 1 \Longrightarrow \Big[c \cdot \arcsin x\Big]_{-1}^{1} = 1 \Longrightarrow c = 1/\pi.$

(b) For $-1 < x < 1$,

$$F(x) = \int_{-1}^{x} \frac{1}{\pi\sqrt{1-x^2}}\,dx = \frac{1}{\pi}\arcsin x + \frac{1}{2}.$$

Thus

$$F(x) = \begin{cases} 0 & x < -1 \\ \frac{1}{\pi}\arcsin x + \frac{1}{2} & -1 \le x < 1 \\ 1 & x \ge 1. \end{cases}$$

7. **(a)** Let F be the distribution function of X. Then X is symmetric about α if and only if for all x, $1 - F(\alpha + x) = F(\alpha - x)$, or upon differentiation $f(\alpha + x) = f(\alpha - x)$.
(b) $f(\alpha + x) = f(\alpha - x)$ if and only if $(\alpha - x - 3)^2 = (\alpha + x - 3)^2$. This is true for all x, if and only if $\alpha - x - 3 = -(\alpha + x - 3)$ which gives $\alpha = 3$. A similar argument shows that g is symmetric about $\alpha = 1$.

4. The set of possible values of X is $A = (0, \infty)$. Let $h\colon (0, \infty) \to \mathbf{R}$ be defined by $h(x) = \log_2 x$. The set of possible values of h is $B = (-\infty, \infty)$. h is invertible and its inverse is $g(y) = 2^y$, where $g'(y) = (\ln 2)2^y$. Thus

$$f_Y(y) = 3e^{-3(2^y)}\left|(\ln 2)2^y\right| = (3\ln 2)2^y e^{-3(2^y)}, \quad y \in (-\infty, \infty).$$

8. Let G and g be distribution and density functions of Y, respectively. Then

$$\begin{aligned} G(t) &= P(Y \le t) = P(Y \le t \mid X \le 1)P(X \le 1) + P(Y \le t \mid X > 1)P(X > 1) \\ &= P(X \le t \mid X \le 1)P(X \le 1) + P\Big(X \ge \frac{1}{t} \;\Big|\; X > 1\Big)P(X > 1). \end{aligned}$$

For $t \ge 1$, this gives

$$G(t) = 1 \cdot \int_0^1 e^{-x}\,dx + 1 \cdot \int_1^{\infty} e^{-x}\,dx = 1.$$

For $0 < t < 1$, this gives

$$G(t) = P(X \le t) + P\Big(X \ge \frac{1}{t}\Big) = \int_0^t e^{-x}\,dx + \int_{1/t}^{\infty} e^{-x}\,dx = 1 - e^{-t} + e^{-1/t}.$$

Hence

$$G(t) = \begin{cases} 0 & t \le 0 \\ 1 - e^{-t} + e^{-1/t} & 0 < t < 1 \\ 1 & t \ge 1. \end{cases}$$

Therefore,

$$g(t) = G'(t) = \begin{cases} e^{-t} + \dfrac{1}{t^2} e^{-1/t} & 0 < t < 1 \\ 0 & \text{elsewhere.} \end{cases}$$

5. $E(X) = \displaystyle\int_{-1}^{1} \frac{x}{\pi\sqrt{1-x^2}}\,dx = 0$, because the integrand is an odd function.

$-\infty$

8. $E(\ln X) = \displaystyle\int_1^2 \frac{2\ln x}{x^2}\,dx$. To calculate this integral, let $U = \ln x$, $dV = 1/x^2$, and use integration by parts:

$$\int_1^2 \frac{2\ln x}{x^2}\,dx = -\frac{2\ln x}{x}\bigg|_1^2 - \int_1^2 -\frac{2}{x^2}\,dx = 1 - \ln 2 = 0.3069.$$

2. $E(X) = \displaystyle\int_1^\infty x \cdot \frac{2}{x^3}\,dx = \int_1^\infty \frac{2}{x^2}\,dx = -\frac{2}{x}\bigg|_1^\infty = 2,$

$E(X^2) = \displaystyle\int_1^\infty x^2 \cdot \frac{2}{x^3}\,dx = 2\ln x\bigg|_1^\infty = \infty$. So $\text{Var}(X)$ does not exist.

12. Since $Y \ge 0$, $P(Y \le t) = 0$ for $t < 0$. For $t \ge 0$,

$$\begin{aligned} P(Y \le t) &= P(|X| \le t) = P(-t \le X \le t) = P(X \le t) - P(X < -t) \\ &= P(X \le t) - P(X \le -t) = F(t) - F(-t). \end{aligned}$$

Hence G, the probability distribution function of $|X|$ is given by

$$G(t) = \begin{cases} F(t) - F(-t) & \text{if } t \ge 0 \\ 0 & \text{if } t < 0; \end{cases}$$

g, the probability density function of $|X|$ is obtained by differentiating G:

$$g(t) = G'(t) = \begin{cases} f(t) + f(-t) & \text{if } t \ge 0 \\ 0 & \text{if } t < 0. \end{cases}$$