Assignment 6

1. $(23-20)/(27-20) = 3/7$.

5. The probability density function of R, the radius of the sphere is

$$f(r) = \begin{cases} \dfrac{1}{4-2} = \dfrac{1}{2} & 2 < r < 4 \\ 0 & \text{elsewhere.} \end{cases}$$

Thus

$$E(V) = \int_2^4 \Big(\frac{4}{3}\pi r^3\Big)\frac{1}{2}\,dr = 40\pi.$$

$$P\Big(\frac{4}{3}\pi R^3 < 36\pi\Big) = P(R^3 < 27) = P(R < 3) = \frac{1}{2}.$$

6. The problem is equivalent to choosing a random number X from $(0, \ell)$. The desired probability is

$$P\Big(X \le \frac{\ell}{3}\Big) + P\Big(X \ge \frac{2\ell}{3}\Big) = \frac{\ell/3}{\ell} + \frac{\ell - (2\ell/3)}{\ell} = \frac{2}{3}.$$

2. $np = 1095/365 = 3$ and $\sqrt{np(1-p)} = \sqrt{3\Big(\dfrac{364}{365}\Big)} = 1.73$. Therefore,

$$P(X \ge 5.5) = P\Big(Z \ge \frac{5.5-3}{1.73}\Big) = 1 - \Phi(1.45) = 0.0735.$$

4. Let

$$g(x) = P(x < Z < x + \alpha) = \frac{1}{\sqrt{2\pi}}\int_x^{x+\alpha} e^{-y^2/2}\,dy.$$

The number x that maximizes $P(x < Z < x + \alpha)$ is the root of $g'(x) = 0$; that is, it is the solution of

$$g'(x) = \frac{1}{\sqrt{2\pi}}\big[e^{-(x+\alpha)^2/2} - e^{-x^2/2}\big] = 0,$$

which is $x = -\alpha/2$.

6. **(a)** $P(X > 35.5) = P\Big(\dfrac{X - 35.5}{4.8} > \dfrac{35.5 - 35.5}{4.8}\Big) = 1 - \Phi(0) = 0.5.$

(b) The desired probability is given by

$$P(30 < X < 40) = P\Big(\frac{30-35.5}{4.8} < X < \frac{40-35.5}{4.8}\Big) = \Phi(0.94) - \Phi(-1.15)$$
$$= \Phi(0.94) + \Phi(1.15) - 1 = 0.8264 + 0.8749 - 1 = 0.701.$$

24. For $t \geq 0$,

$$P(Y \leq t) = P\big(-\sqrt{t} \leq X \leq \sqrt{t}\big) = P\Big(-\frac{\sqrt{t}}{\sigma} \leq Z \leq \frac{\sqrt{t}}{\sigma}\Big) = 2\Phi\Big(\frac{\sqrt{t}}{\sigma}\Big) - 1.$$

Let f be the probability density function of Y. Then

$$f(t) = \frac{d}{dt}P(Y \leq t) = 2\frac{1}{2\sigma\sqrt{t}}\Phi'\Big(\frac{\sqrt{t}}{\sigma}\Big), \quad t \geq 0.$$

So

$$f(t) = \begin{cases} \dfrac{1}{\sigma\sqrt{2\pi t}}\exp\Big(-\dfrac{t}{2\sigma^2}\Big) & t \geq 0 \\ 0 & t \leq 0. \end{cases}$$

2. Let X be the weight of a randomly selected women from this community. The desired quantity is

$$P(X > 170 \mid X > 140) = \frac{P(X > 170)}{P(X > 140)} = \frac{P\Big(Z > \dfrac{170-130}{20}\Big)}{P\Big(Z > \dfrac{140-130}{20}\Big)}$$

$$= \frac{P(Z > 2)}{P(Z > 0.5)} = \frac{1-\Phi(2)}{1-\Phi(0.5)} = \frac{1-0.9772}{1-0.6915} = 0.074.$$