Assignment 8

6. The following table gives $p(x, y)$, the joint probability mass function of X and Y; $p_X(x)$, the marginal probability mass function of X; and $p_Y(y)$, the marginal probability mass function of Y.

x	y = 0	1	2	3	4	5	$p_X(x)$
2	1/36	0	0	0	0	0	1/36
3	0	2/36	0	0	0	0	2/36
4	1/36	0	2/36	0	0	0	3/36
5	0	2/36	0	2/36	0	0	4/36
6	1/36	0	2/36	0	2/36	0	5/36
7	0	2/36	0	2/36	0	2/36	6/36
8	1/36	0	2/36	0	2/36	0	5/36
9	0	2/36	0	2/36	0	0	4/36
10	1/36	0	2/36	0	0	0	3/36
11	0	2/36	0	0	0	0	2/36
12	1/36	0	0	0	0	0	1/36
$p_Y(y)$	6/36	10/36	8/36	6/36	4/36	2/36	

7. $p(1, 1) = 0$, $p(1, 0) = 0.30$, $p(0, 1) = 0.50$, $p(0, 0) = 0.20$.

9. (a) $f_X(x) = \int_0^x 2\,dy = 2x, \quad 0 \le x \le 1; \quad f_Y(y) = \int_y^1 2\,dx = 2(1-y), \quad 0 \le y \le 1.$

(b) $$E(X) = \int_0^1 x f_X(x)\,dx = \int_0^1 x(2x)\,dx = 2/3;$$

$$E(Y) = \int_0^1 y f_Y(y)\,dy = \int_0^1 2y(1-y)\,dy = 1/3.$$

(c) $$P\left(X < \frac{1}{2}\right) = \int_0^{1/2} f_X(x)\,dx = \int_0^{1/2} 2x\,dx = \frac{1}{4},$$

$$P(X < 2Y) = \int_0^1 \int_{x/2}^x 2\,dy\,dx = \frac{1}{2},$$

$$P(X = Y) = 0.$$

11. $f_X(x) = \int_0^2 \frac{1}{2} y e^{-x}\,dy = e^{-x}, \quad x > 0; \quad f_Y(y) = \int_0^\infty \frac{1}{2} y e^{-x}\,dx = \frac{1}{2} y, \quad 0 < y < 2.$

8. For $i,\ j \in \{0, 1, 2, 3\}$, the sum of the numbers in the ith row is $p_X(i)$ and the sum of the numbers in the jth row is $p_Y(j)$. We have that

$$p_X(0) = 0.41, \qquad p_X(1) = 0.44, \qquad p_X(2) = 0.14, \qquad p_X(3) = 0.01;$$
$$p_Y(0) = 0.41, \qquad p_Y(1) = 0.44, \qquad p_Y(2) = 0.14, \qquad p_Y(3) = 0.01.$$

Since for all $x, y \in \{0, 1, 2, 3\}$, $p(x, y) = p_X(x)p_Y(y)$, X and Y are independent.

11. We have that

$$f_X(x) = \int_0^\infty x^2 e^{-x(y+1)}\,dy = xe^{-x}, \quad x \ge 0;$$
$$f_Y(y) = \int_0^\infty x^2 e^{-x(y+1)}\,dx = \frac{2}{(y+1)^3}, \quad y \ge 0,$$

where the second integral is calculated by applying integration by parts twice. Now since $f(x, y) \neq f_X(x)f_Y(y)$, X and Y are not independent.

23. Note that

$$f_X(x) = \int_{-\infty}^{\infty} g(x)h(y)\,dy = g(x)\int_{-\infty}^{\infty} h(y)\,dy,$$
$$f_Y(y) = \int_{-\infty}^{\infty} g(x)h(y)\,dx = h(y)\int_{-\infty}^{\infty} g(x)\,dx.$$

Now

$$\begin{aligned} f_X(x)f_Y(y) &= g(x)h(y)\int_{-\infty}^{\infty} h(y)\,dy \int_{-\infty}^{\infty} g(x)\,dx \\ &= f(x, y)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(y)g(x)\,dy\,dx \\ &= f(x, y)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x, y)\,dy\,dx = f(x, y). \end{aligned}$$

This relation shows that X and Y are independent.

10. (a) $\displaystyle\int_0^\infty \int_{-x}^{x} ce^{-x}\,dy\,dx = 1$ implies that $c = 1/2$.

(b) $$f_{X|Y}(x|y) = \frac{f(x, y)}{f_Y(y)} = \frac{(1/2)e^{-x}}{\displaystyle\int_{|y|}^{\infty} (1/2)e^{-x}\,dx} = e^{-x+|y|}, \quad x > |y|,$$

$$f_{Y|X}(y|x) = \frac{(1/2)e^{-x}}{\displaystyle\int_{-x}^{x} (1/2)e^{-x}\,dy} = \frac{1}{2x}, \quad -x < y < x.$$

(c) By part (b), given $X = x$, Y is a uniform random variable over $(-x, x)$. Therefore, $E(Y|X = x) = 0$ and

$$\text{Var}(Y|X = x) = \frac{\big[x - (-x)\big]^2}{12} = \frac{x^2}{3}.$$