12. In the experiment of tossing a coin, let A be the event of obtaining heads and B be the event of obtaining tails.

13. **(a)** $P(A \cup B) \geq P(A) = 1$, so $P(A \cup B) = 1$. Now

$$1 = P(A \cup B) = P(A) + P(B) - P(AB) = 1 + P(B) - P(AB)$$

gives $P(B) = P(AB)$.

(b) If $P(A) = 0$, then $P(AB) = 0$; so $P(AB) = P(A)P(B)$ is valid. If $P(A) = 1$, by part (a), $P(AB) = P(B) = P(A)P(B)$.

15. $P(AB) = P(A)P(B)$ implies that $P(A) = P(A)P(B)$. This gives $P(A)\big[1 - P(B)\big] = 0$; so $P(A) = 0$ or $P(B) = 1$.

29. Let E_i be the event that the switch located at i is closed. The desired probability is

$$P(E_1E_2E_4E_6 \cup E_1E_3E_5E_6) = P(E_1E_2E_4E_6) + P(E_1E_3E_5E_6) - P(E_1E_2E_3E_4E_5E_6) = 2p^4 - p^6.$$

7. **(a)** $\sum_{k=1}^{5} kx = 1 \Rightarrow k = 1/15$.

(b) $k(-1)^2 + k + 4k + 9k = 1 \Rightarrow k = 1/15$.

(c) $$\sum_{x=1}^{\infty} k\Big(\frac{1}{9}\Big)^x = 1 \Rightarrow k = \frac{1}{\sum_{x=1}^{\infty}(1/9)^x} = 1\Big/\Big[\frac{1/9}{1-(1/9)}\Big] = 8.$$

(d) $$k(1 + 2 + \cdots + n) = 1 \Rightarrow k = \frac{1}{\big[n(n+1)\big]/2} = \frac{2}{n(n+1)}.$$

(e) $$k(1^2 + 2^2 + \cdots + n^2) = 1 \Rightarrow k = \frac{6}{n(n+1)(2n+1)}.$$

14. For $i = 0, 1, 2$, and 3, we have

$$P(X = i) = \frac{\dbinom{10}{i}\dbinom{10-i}{6-2i}2^{6-2i}}{\dbinom{20}{6}}.$$

The numerical values of these probabilities are as follows.

i	0	1	2	3
$p(i)$	112/323	168/323	42/323	1/323

5. Let X be the net gain in one play of the game. The set of possible values of X is $\{-8, -4, 0, 6, 10\}$. The probabilities associated with these values are

$$p(-8) = p(0) = \frac{1}{\binom{5}{2}} = \frac{1}{10}, \quad p(-4) = \frac{\binom{2}{1}\binom{2}{1}}{\binom{5}{2}} = \frac{4}{10},$$

and $p(6) = p(10) = \dfrac{\binom{2}{1}}{\binom{5}{2}} = \dfrac{2}{10}$. Hence

$$E(X) = -8 \cdot \frac{1}{10} - 4 \cdot \frac{4}{10} + 0 \cdot \frac{1}{10} + 6 \cdot \frac{2}{10} + 10 \cdot \frac{2}{10} = \frac{4}{5}.$$

Since $E(X) > 0$, the game is not fair.

12. $E(X) = \sum_{i=1}^{10} i \cdot \dfrac{1}{10} = \dfrac{11}{2}$ and $E(X^2) = \sum_{i=1}^{10} i^2 \cdot \dfrac{1}{10} = \dfrac{77}{2}$. So

$$E\big[X(11 - X)\big] = E(11X - X^2) = 11 \cdot \frac{11}{2} - \frac{77}{2} = 22.$$

3. $E(X) = \sum_{x=-3}^{3} x p(x) = -1$, $E(X^2) = \sum_{x=-3}^{3} x^2 p(x) = 4$. Therefore, $\text{Var}(X) = 4 - 1 = 3$.

4. p, the probability mass function of X is given by

x	-3	0	6
$p(x)$	3/8	3/8	2/8

Thus

$$E(X) = -\frac{9}{8} + \frac{12}{8} = \frac{3}{8}, \qquad E(X^2) = \frac{27}{8} + \frac{72}{8} = \frac{99}{8},$$

$$\text{Var}(X) = \frac{99}{8} - \frac{9}{64} = \frac{783}{64} = 12.234, \qquad \sigma_X = \sqrt{12.234} = 3.498.$$