Ma 612 Mathematical Statistics
Final Examination

due Tuesday May 15, 2012, by 5:00pm

(1) Let $X_1, \ldots, X_n$ be iid random variables with:

$$P(X_j = 1) = \theta = 1 - P(X_j = 0),$$

with n a fixed constant and $0 < \theta < 1$.

(a) Let T_k denote the uniform minimum variance unbiased estimator (UMVUE) of θ^k. Give an explicit formula for T_k, for $k = 1, 2, \ldots, n$.

(b) Suppose we are interested in estimating the odds ratio $r = \frac{\theta}{1-\theta}$. Note that r does not have an unbiased estimator (you do not have to show this). What is the MLE of r? What is the bias of this MLE?

(2) Let $X_1, \ldots, X_n$ be a random sample from the density:

$$f(x|\theta) = e^{-(x-\theta)} \mathbf{1}_{\{x>\theta\}}$$

(a) Show that the family has a monotone likelihood ratio and identify the statistic $T(x)$.

(b) Determine the uniform most powerful test (UMP) of testing

$$\begin{cases} H_0: & \theta \le \theta_0 \\ H_a: & \theta > \theta_0 \end{cases}$$

Show how to select a constant C such that the test has level α.

(3) Let X $X_1, \ldots, X_n$ be a random sample from the exponential distribution $f(x|\theta) = \frac{1}{\theta} e^{-\frac{x}{\theta}}$, $x > 0$, $\theta > 0$. Find:

$$\mathbf{E}\left[\frac{\sum_{i=1}^n iX_i}{\sum_{i=1}^n X_i}\right]$$

(Hint: Basu)

(4) Do 8.55 on page 413 from your textbook. (please note if you reproduce the solution manual you will earn negative points).

(5) Let X be a discrete random variable with the pdf:

$$f(x|\theta) = \binom{r+x-1}{x}\theta^x(1+\theta)^{-(r+x)}, x = 0, 1, 2, \ldots$$

where r is a known positive integer and θ is a parameter[1]. You can calculate the mean and variance of X as $\mathbf{E}(X) = r\theta$ and $Var(X) = r\theta(1+\theta)$.

We are interested in estimating θ using the loss function:

$$L(\theta, \delta) = \frac{(\theta - \delta)^2}{\theta(1+\theta)}.$$

For this purpose we assume that the prior distribution on θ is:

$$\pi(\theta) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\theta^{\alpha-1}(1+\theta)^{-(\alpha+\beta)}\mathbf{1}_{\{\theta>0\}}$$

(a) Find the Bayes estimator δ^π of θ with respect to the above prior distribution.

(b) Find the risk function $R_{\delta^\pi}(\theta) = \mathbf{E}\left[L(\theta, \delta^\pi(X))\right]$ of the Bayes estimator found in (a).

(6) Let $X_1, \ldots, X_n$ be a random sample from the distribution

$$f(x|\theta) = \frac{3\theta^3}{x^4}\mathbf{1}_{\{x>\theta\}}.$$

Find the method of moments estimator of θ, prove its consistency and obtain its limiting distribution as $n \to \infty$.

[1]This is the Negative Binomial distribution reparametrized with $\theta = (1-p)/p$, where p is the probability of success on an individual trial