Ma 221 Homework Solutions Due Date: January 22-23, 2009

2.2 pg. 46 # 1, 4, 5, 7, 9, 11, 15, 17, 19, 21, 22, 23, 25
(**Underlined problems are not handed in**)

In problems 1, 4 and 5, determine whether the given differential equation is separable.

1) $\frac{dy}{dx} = 2y^3 + y + 4 \Rightarrow \quad dx\left(\frac{dy}{dx}\right) = (2y^3 + y + 4)dx \Rightarrow \quad \frac{dy}{2y^3+y+4} = dx$
therefore, this equation is separable.

4.)
$\frac{ds}{dt} = t\ln(s^{2t}) + 8t^2 \Rightarrow \quad \frac{ds}{dt} = 2t^2 \ln s + 8t^2 \Rightarrow \quad \frac{ds}{dt} = 2t^2(\ln s + 4) \Rightarrow \frac{ds}{\ln s+4} = 2t^2 dt$
therefore, this equation is separable.

5.) $s^2 + \frac{ds}{dt} = \frac{s+1}{st}$
Writing the equation in the form

$$\frac{ds}{dt} = \frac{s+1}{st} - s^2$$

shows that the equation is not separable.

In problems 7, 9, 11 and 15, solve the equation.

7.) $\frac{dy}{dx} = \frac{1-x^2}{y^2}$

$\frac{dy}{dx} = \frac{1-x^2}{y^2} \Rightarrow \quad y^2 dy = (1-x^2)dx \Rightarrow \quad \int y^2 dy = \int(1-x^2)dx \Rightarrow \quad \frac{1}{3}y^3 =$
$\Rightarrow \quad y = \sqrt[3]{3x - x^3 + C}$

9) $\frac{dy}{dx} = y(2+\sin x)$

$\frac{dy}{dx} = y(2+\sin x) \Rightarrow \quad \frac{dy}{y} = (2+\sin x)dx \Rightarrow \quad \int \frac{dy}{y} = \int(2+\sin x)dx \Rightarrow$
$\Rightarrow \quad y = e^{2x-\cos x+C} = Ce^{2x-\cos x}$

11) $\frac{dy}{dx} = \frac{\sec^2 y}{1+x^2}$

$\frac{dy}{dx} = \frac{\sec^2 y}{1+x^2} \Rightarrow \quad \frac{dy}{\sec^2 y} = \frac{dx}{1+x^2}$

Using trigonemtric identities we have:

$\Rightarrow \sec y = \frac{1}{\cos y}$ and $\cos^2 y = \frac{1}{2}(1+\cos 2y)$

$$\Rightarrow \frac{dy}{\sec^2 y} = \frac{dx}{(1+x^2)} \Rightarrow \quad \frac{(1+\cos 2y)dy}{2} = \frac{dx}{1+x^2} \Rightarrow \quad \int \frac{(1+\cos 2y)dy}{2} = \int \frac{dx}{1+x^2}$$

$$\Rightarrow \quad \frac{1}{2}\left(y + \frac{1}{2}\sin 2y\right) = \arctan x + C_1 \Rightarrow \quad 2y + \sin 2y = 4\arctan x + 4C_1$$

$$\Rightarrow \quad 2y + \sin 2y = 4\arctan x + C$$

15.) $y^{-1}dy + ye^{\cos x}\sin x dx = 0$

$$-ye^{\cos x}\sin x dx = y^{-1}dy$$

$$-\int e^{\cos x}\sin x dx = \int y^{-2}dy$$

Substituting:

$$\text{let } u = \cos x$$

$$du = -\sin x dx$$

$$-du = \sin x dx$$

$$\int e^u du = \int y^{-2}dy$$

$$e^u = -y^{-1} + C$$

$$e^{\cos x} = y^{-1} + C$$

$$y = \frac{1}{C - e^{\cos x}}$$

In problems 17, 19, 21, 22, 23, and 25, solve the initial value problem.

17) $y' = x^3(1-y), \quad y(0) = 3$

$$\frac{dy}{dx} = x^3(1-y) \Rightarrow \quad \frac{dy}{(1-y)} = x^3 dx \Rightarrow \quad \int \frac{dy}{(1-y)} = \int x^3 dx \Rightarrow \quad -\ln|1-y| +$$

$$|1-y| = \exp\left(C_1 - \frac{x^4}{4}\right) = Ce^{-\frac{x^4}{4}}$$

Substituting the IC $y(0) = 3$

$|1-3| = Ce^{-\frac{0^4}{4}} \Rightarrow \quad |-2| = C = 2$

$\Rightarrow |1-y| = 2e^{-\frac{x^4}{4}}$

Since $1 - y(0) = 1 - 3 < 0$, on an interval containing $x = 0$ one has $1 - y(x) < 0$ and so $|1 - y(x)| = y(x) - 1$. The solution is then:

$\Rightarrow y - 1 = 2e^{-\frac{x^4}{4}} = y = 2e^{-\frac{x^4}{4}} + 1$

19.) $\frac{dy}{d\theta} = y\sin\theta, \quad y(\pi) = -3$

$\frac{dy}{d\theta} = y\sin\theta \Rightarrow \quad \frac{dy}{y} = \sin\theta d\theta \Rightarrow \quad \int \frac{dy}{y} = \int \sin\theta d\theta \Rightarrow \quad \ln|y| = -\cos\theta + C_1$

$\Rightarrow \quad |y| = e^{-\cos\theta + C_1} \Rightarrow \quad y = -Ce^{-\cos\theta}$

because at the initial point, $\theta = \pi, \quad y(\pi) = -3 < 0$

$-3 = y(\pi) = -Ce^{-\cos\pi} = -Ce^{1} \Rightarrow \quad C = 3e^{-1} \Rightarrow \quad y = -3e^{-1}e^{-\cos\theta} = -3e^{-1-\cos\theta}$

21) $\frac{dy}{dx} = 2\sqrt{y+1}\cos x, \quad y(\pi) = 0$

$\frac{dy}{2\sqrt{y+1}} = \cos x dx = \frac{1}{2}(y+1)^{-1/2}dy = \cos x dx \Rightarrow \quad \int \frac{1}{2}(y+1)^{-1/2}dy = \int \cos x dx$

$(y+1)^{1/2} = \sin x + C$ Substituting the IC

$y(\pi) = 0 \Rightarrow \quad (0+1)^{1/2} = \sin\pi + C \Rightarrow \quad C = 1$

$(y+1)^{1/2} = \sin x + 1 \Rightarrow \quad y = (\sin x + 1)^2 - 1 = \sin^2 x + 2\sin x$

22) $x^2dx + 2ydy = 0, \quad y(0) = 2$

$x^2dx = -2ydy \Rightarrow \quad \int x^2dx = \int -2ydy \Rightarrow \quad \frac{x^3}{3} = -y^2 + C$ Substituting the IC $y(0) = 2$:

$\frac{0^3}{3} = -2^2 + C \Rightarrow \quad C = 4$

$\frac{x^3}{3} = -y^2 + 4 \Rightarrow \quad y = \sqrt{4 - \frac{x^3}{3}}$

23.) $\frac{dy}{dx} = 2x\cos^2 y\,, \quad y(0) = \frac{\pi}{4}$

$$\frac{dy}{\cos^2 y} = 2xdx$$

$$\int \sec^2 y dy = \int 2xdx$$

$$\tan y = x^2 + C$$

$$\tan\left(\frac{\pi}{4}\right) = C$$

$$C = 1$$

Thus

$$\tan y = x^2 + 1$$

or

$$y = \arctan(x^2 + 1)$$

25.) $\frac{dy}{dx} = x^2(1+y) \qquad y(0) = 3$

$$\frac{dy}{dx} = x^2(1+y) \Rightarrow \qquad \int \frac{dy}{1+y} = \int x^2 dx \Rightarrow \qquad \ln|1+y| = \frac{1}{3}x^3 + C_1 \Rightarrow \qquad 1+y = e^{\frac{1}{3}x^3 + C_1}$$

$$\Rightarrow \qquad y = Ce^{\frac{1}{3}x^3} - 1$$

$$3 = y(0) = C - 1 \Rightarrow \qquad C = 4 \Rightarrow \qquad y = 4e^{\frac{1}{3}x^3} - 1$$

2.3 p.54 to p. 55 #1, 3, 5, 7, 10, 11, 13, 15, 17, 18, 19, 21, 22, 30 (Underlined Problems are handed in)

For problems 1, 3, and 5 determine whether the given equation is separable, linear, neither, or both.

1.) $x^2 \frac{dy}{dx} + \cos x = y$

Isolating $\frac{dy}{dx}$ we get:
$\frac{dy}{dx} = \frac{y - \cos x}{x^2}$. Since the right hand side cannot be represented as a product $g(x)p(y)$, the equation is **not seperable**.

3.) $x\frac{dx}{dt} + t^2 x = \sin t$
In this equation, the independent variable is t and the dependent variable is x. Dividing by x we obtain
$\frac{dx}{dt} = \frac{\sin t}{x} - t^2$.
Therefore, it is neither linear, because of the $\frac{\sin t}{x}$ term, nor separable, because the righthand side is not a product of functions of single variables x and t.

5.) $(t^2+1)\frac{dy}{dt} = yt - y$

This is a linear equation with independent variable t and dependent variable y. This is a **seperable** equation as shown:

Isolating $\frac{dy}{dt}$ we get:

$$\frac{dy}{dt} = \frac{y(t-1)}{(t^2+1)} \Rightarrow \qquad \frac{dy}{y} = \frac{(t-1)}{(t^2+1)}dt \Rightarrow \qquad \frac{dy}{dt} = \frac{(t-1)}{(t^2+1)}y = g(t)p(y)$$

For problems 7, 10, 11, 13 and 15, obtain the general solution to the equation

7.) $\frac{dy}{dx} - y = e^{3x}$
In this equation, $P(x) = -1$ and $Q(x) = e^{3x}$
The integrating factor is thus: $\mu(x) = \exp(\int P(x)dx) = \exp(\int(-1)dx) = e^{-x}$
Multiplying both sides of the equation by $\mu(x)$ and integrating yields:

$$e^{-x}\frac{dy}{dx} - e^{-x}y = e^{-x}e^{3x} = e^{2x} \Rightarrow \frac{d(e^{-x}y)}{dx} = e^{2x}$$

$\Rightarrow\ e^{-x}y = \int e^{2x}dx = \frac{1}{2}e^{2x} + C \Rightarrow \boxed{y = (\frac{1}{2}e^{2x} + C)e^{x} = \frac{e^{3x}}{2} + Ce^{x}}$

10.) $x\frac{dy}{dx} + 2y = x^{-3}$
Putting the equation in standard form:
$\frac{dy}{dx} + \frac{2}{x}y = x^{-4}$
Find $\mu(x) = e^{\int Pdx} = e^{\int (\frac{2}{x})dx} = e^{(2\ln|x|)} = (|x|)^2 = x^2$
Multiply through by $\mu(x)$ to get
$x^2\frac{dy}{dx} + 2xy = x^{-2} \Rightarrow \qquad \frac{d}{dx}yx^2 = x^{-2}$
Integrate to get
$\int \frac{d}{dx}yx^2 = \int x^{-2}$
$yx^2 = -x^{-1} + C$
Solve explicitly for y
$\Rightarrow \boxed{y = -x^{-3} + Cx^{-2}}$

11.) $(t + y + 1)dt - dy = 0$
Choosing t as the independent variable and y as the dependent variable, the equation can be put into standard form:

$t + y + 1 - \frac{dy}{dt} = 0 \Rightarrow \frac{dy}{dt} - y = t + 1$

Thus: $P(t) = -1$ and $\mu(t) = \exp\left[\int(-1)dt\right] = e^{-t}$
Multiplying both sides by $\mu(t)$ and integrating yields:
$e^{-t}\frac{dy}{dt} - e^{-t}y = (t+1)e^{-t} \Rightarrow \frac{d(e^{-t}y)}{dt} = (t+1)e^{-t}$
$\Rightarrow\ e^{-t}y = \int(t+1)e^{-t}dt = -(t+1)e^{-t} + \int e^{-t}dt = -(t+1)e^{-t} - e^{-t} + C = -(t+2)e^{-t} + C$
$\Rightarrow \boxed{y = e^{t}(-(t+2)e^{-t} + C) = -t - 2 + Ce^{t}}$

13.) $y\frac{dx}{dy} + 2x = 5y^3$
In this problem, the independent variable is y and the dependent variable is x. So, we divide the equation by y to rewrite it in standard form.

$y\frac{dx}{dy} + 2x = 5y^3 \Rightarrow \qquad \frac{dx}{dy} + \frac{2}{y}x = 5y^2$

Therefore, $P(y) = \frac{2}{y}$ and the integrating factor, $\mu(y)$, is

$$\mu(y) = e^{\int \frac{2}{y}dy} = e^{2\ln|y|} = |y|^2 = y^2$$

Multiplying the equation (in standard form) by y^2 and inegrating yield

$\left(\frac{dx}{dy} + \frac{2}{y}x = 5y^2\right)y^2 = y^2\frac{dx}{dy} + 2yx = 5y^4 \Rightarrow \qquad \frac{d}{dy}(y^2x) = 5y^4$
$\Rightarrow \qquad y^2x = \int 5y^4dy = y^5 + C$
$\Rightarrow \boxed{x = y^{-2}(y^5 + C) = y^3 + Cy^{-2}}$

15.) $(x^2+1)\frac{dy}{dx}+xy = x$
Divide by (x^2+1)
$\frac{dy}{dx}+\frac{x}{x^2+1}y = \frac{x}{x^2+1}$
so $P(x) = \frac{x}{x^2+1}$
Find $\mu(x) = e^{\int P(x)dx}$
$\mu(x) = e^{\int \frac{x}{x^2+1}} = e^{\frac{1}{2}\ln(x^2+1)} = (x^2+1)^{(1/2)}$
Multiply through by $\mu(x)$ to get
$(x^2+1)^{\frac{1}{2}}y' + \frac{x}{(x^2+1)^{\frac{1}{2}}}y = \frac{x}{(x^2+1)^{\frac{1}{2}}}$ or
$\frac{d}{dx}\left((x^2+1)^{\frac{1}{2}}y\right) = \frac{x}{(x^2+1)^{\frac{1}{2}}}$
Integrating gives
$y(x^2+1)^{(1/2)} = (x^2+1)^{(1/2)}+C$
Solve explicitly for y
$y = 1+C(x^2+1)^{-(1/2)}$

For problems 17, 18 and 19, 21, 22 solve the initial value problems.

17.) $\frac{dy}{dx}-\frac{y}{x} = xe^x \qquad y(1) = e-1$
This is a linear equation with $P(x) = -1/x$ and $Q(x) = xe^x$. The integrating factor is given by:
$\mu(x) = e^{\int Pdx} = e^{\int -\frac{1}{x}dx} = e^{-\ln x} = \frac{1}{x}; \qquad$ For $x > 0$
Multiply through by $\mu(x)$ to get
$\frac{1}{x}y' - \frac{y}{x^2} = e^x$
or
$\frac{1}{x}\frac{dy}{dx} - \frac{y}{x^2} = e^x$
Integrate to get
$\frac{y}{x} = e^x + C$
Solve explicitly for y
$y = xe^x + Cx$
Plug in initial condition $y(1) = e-1$ and solve for C
$e-1 = e+C \Rightarrow \qquad C = -1$
Plug in the value for C
$y = xe^x - x$
Using SNB to check our answer we have
$\begin{matrix} \frac{dy}{dx}-\frac{y}{x} = xe^x \\ y(1) = e-1 \end{matrix}$, Exact solution is: $xe^x - x$

18.) $\frac{dy}{dx}+4y-e^{-x} = 0$; $y(0) = \frac{4}{3}$
$\frac{dy}{dx}+4y = e^{-x}$
Find $\mu(x) = e^{\int Pdx} = e^{\int 4dx} = e^{4x}$

Multiply through by $\mu(x)$ to get
$e^{4x}y' + 4e^{4x}y = e^{3x}$
or
$\frac{d}{dx}(e^{4x}y) = e^{3x}$
Integrate to get
$ye^{4x} = \frac{1}{3}e^{3x} + c$
Solve explicitly for y
$y = \frac{1}{3}e^{-x} + \frac{C}{e^{4x}}$
Plug in initial condition $y(0) = \frac{4}{3}$ and solve for C
$\frac{4}{3} = \frac{1}{3}e^{-0} + \frac{C}{e^{4(0)}}$
$\frac{4}{3} = \frac{1}{3} + C$
So $C = 1$
Plug in the value for C
$y = \frac{1}{3}e^{-x} + e^{-4x}$
Using SNB to check our answer we have

$$\begin{array}{c} \frac{dy}{dx} + 4y - e^{-x} = 0 \\ y(0) = \frac{4}{3} \end{array}, \text{ Exact solution is: } y(x) = \frac{1}{3}e^{-x} + e^{-4x}$$

19.) $t^3\frac{dx}{dt} + 3t^2x = t, \qquad x(2) = 0$
In this problem, t is the dependent variable and x is the dependent variable. Notice the left side of the equation is the derivative of xt^3 with respect to t. Using the product rule for differentiation yields:
$\frac{d(xt^3)}{dt} = \frac{dx}{dt}t^3 + x\frac{d(t^3)}{dt} = t^3\frac{dx}{dt} + 3t^2x$
The equation becomes:
$\frac{d}{dt}(xt^3) = t \quad \Rightarrow \quad xt^3 = \int t\,dt = \frac{t^2}{2} + C$
$\Rightarrow x = t^{-3}(\frac{t^2}{2} + C) = \frac{1}{2t} + \frac{C}{t^3} \Rightarrow \boxed{x = \frac{1}{2t} + \frac{C}{t^3}}$
$0 = x(2) = \frac{1}{2(2)} + \frac{C}{2^3} \Rightarrow \quad \frac{1}{4} + \frac{C}{8} = 0 \Rightarrow \quad C = -2 \Rightarrow \quad \boxed{x = \frac{1}{2t} - \frac{2}{t^3}}$

21.) $\cos x\frac{dy}{dx} + y\sin x = 2x\cos^2 x, \qquad y(\frac{\pi}{4}) = \frac{-15\sqrt{2}\pi^2}{32}$
Putting the equation in standard form:
$\frac{dy}{dx} + \frac{\sin x}{\cos x}y = 2x\cos x \Rightarrow \quad \frac{dy}{dx} + (\tan x)y = 2x\cos x$
Find $\mu(x) = e^{\int P dx} = e^{\int \tan x dx} = e^{(-\ln|\cos x|)} = |\cos x|^{-1}$
At the initial point,$x = \frac{\pi}{4}, \cos\frac{\pi}{4} > 0$ therefore we can take $\mu(x) = (\cos x)^{-1}$
Multiply through by $\mu(x)$ to get
$\frac{1}{\cos x}\frac{dy}{dx} + \frac{\sin x}{\cos^2 x}y = 2x \quad \Rightarrow \quad \frac{d}{dx}(\frac{y}{\cos x}) = 2x$
Integrate to get
$\int \frac{d}{dx}(\frac{y}{\cos x}) = \int 2x$
$\frac{y}{\cos x} = x^2 + C$
Solve explicitly for y

$y = x^2 \cos x + C \cos x$

Plug in initial condition $y(\frac{\pi}{4}) = \frac{-15\sqrt{2}\pi^2}{32}$ and solve for C

$\frac{-15\sqrt{2}\pi^2}{32} = \frac{\pi}{4}^2 \cos \frac{\pi}{4} + C \cos \frac{\pi}{4}$

So $C = -\pi^2$

Plug in the value for C

$y = x^2 \cos x - \pi^2 \cos x = \cos x(x^2 - \pi^2)$

22.) $\sin x \frac{dy}{dx} + y \cos x = x \sin x, \qquad y(\frac{\pi}{2}) = 2$

Putting the equation in standard form:

$\frac{dy}{dx} + \frac{\cos x}{\sin x} y = x \Rightarrow \qquad \frac{dy}{dx} + (\cot x)y = x$

Find $\mu(x) = e^{\int P dx} = e^{\int (\frac{\cos x}{\sin x}) dx} = e^{(\ln(\sin x))} = \sin x$

Multiply through by $\mu(x)$ to get

$\sin x \frac{dy}{dx} + y \cos x = x \sin x \Rightarrow \qquad \frac{d}{dx}(y \sin x) = x \sin x$

Integrate to get

$\int \frac{d}{dx}(y \sin x) = \int x \sin x$ (Using integration by parts)

$y \sin x = \sin x - x \cos x + C$

Solve explicitly for y

$y = 1 - x \frac{\cos x}{\sin x} + \frac{C}{\sin x} = 1 - x \cot x + \frac{C}{\sin x}$

Plug in initial condition $y(\frac{\pi}{2}) = 2$ and solve for C

$2 = 1 - \frac{\pi}{2} \frac{\cos \frac{\pi}{2}}{\sin \frac{\pi}{2}} + \frac{C}{\sin \frac{\pi}{2}}$

So $C = 1$

Plug in the value for C

$y = 1 - x \cot x + \frac{1}{\sin x} = 1 - x \cot x + \csc x$

30.) Show that the substitution $v = y^3$ reduces equation $\frac{dy}{dx} + 2y = xy^{-2}$ to the equation $\frac{dv}{dx} + 6v = 3x$. Then solve the equation for v, and make the substitution $v = y^3$ to obtain the solution the equation $\frac{dy}{dx} + 2y = xy^{-2}$.

$\frac{dy}{dx} + 2y = xy^{-2}$

Divide by y^{-2}

$y^2 \frac{dy}{dx} + 2y^3 = x$

$v = y^3$

Differentiate v with respect to x

$\frac{dv}{dx} = 3y^2 \frac{dy}{dx}$

Divide by 3

$\frac{1}{3} \frac{dv}{dx} = y^2 \frac{dy}{dx}$

Notice that $\frac{1}{3} \frac{dv}{dx}$ is equal to the first term on the left hand side of the equation. Make that substitution.

$\frac{1}{3} \frac{dv}{dx} + 2v = x$

Now multiply by 3 to get a first order linear differential equation.

$\frac{dv}{dx} + 6v = 3x$

........(First order linear differential equation)
Find $\mu(x)$
$\mu(x) = e^{\int 6dx} = e^{6x}$
Multiply through by $\mu(x)$ to get
$e^{6x}v' + 6e^{6x}v = \frac{d}{dx}(e^{6x}v) = 3xe^{6x}$Now integrate
$ve^{6x} = \int e^{6x}(3x)dx = \frac{1}{2}xe^{6x} - \frac{1}{12}e^{6x} + C$
Where SNB was used to evaluate the integral.
Solve explicitly for v yields
$v = \frac{1}{2}x - \frac{1}{12} + Ce^{-6x}$
Plug y back into the equation from the substitution $v = y^3$ and solve for y.
$y^3 = \frac{1}{2}x - \frac{1}{12} + Ce^{-6x}$
or
$y = (\frac{1}{2}x - \frac{1}{12} + Ce^{-6x})^{\frac{1}{3}}$
Using SNB and MuPAD to solve the equation we get

$\frac{dy}{dx} + 2y = xy^{-2}$, Exact solution is: $\sqrt[3]{\frac{1}{2}x - 3\frac{C_1}{e^{6x}} - \frac{1}{12}}$, $-\frac{1}{2}\sqrt[3]{\frac{1}{2}x - 3\frac{C_1}{e^{6x}} - \frac{1}{12}} - \frac{1}{2}i\sqrt{3}\sqrt[3]{\frac{1}{2}x - 3\frac{C_1}{e^{6x}} - \frac{1}{12}}$, $\frac{1}{2}i\sqrt{3}\sqrt[3]{\frac{1}{2}x - 3\frac{C_1}{e^{6x}} - \frac{1}{12}} - \frac{1}{2}\sqrt[3]{\frac{1}{2}x - 3\frac{C_1}{e^{6x}} - \frac{1}{12}}$

Using SNB and Maple to solve the equation we get

$\frac{dy}{dx} + 2y = xy^{-2}$, Exact solution is: $y(x) = \frac{1}{6}\sqrt[3]{108x - 18 + 216e^{-6x}C_1}$, $y(x) = -\frac{1}{12}\sqrt[3]{108x - 18 + 216e^{-6x}C_1} + \frac{1}{12}i\sqrt{3}\sqrt[3]{108x - 18 + 216e^{-6x}C_1}$,$y(x) = -\frac{1}{12}\sqrt[3]{108x - 18 + 216e^{-6x}C_1} - \frac{1}{12}i\sqrt{3}\sqrt[3]{108x - 18 + 216e^{-6x}C_1}$

Since $6^3 = 216$ it can be seen that the first answer given by both kernels is the same as our solution. The other answers given by SNB must be the other cube roots of our solution.