

MA 221 Homework Solutions

Due date: Tuesday, January 27, 2009

pg. 65 Sec. 2.4 # 9, 11, 13, 15, 17, 19, 23, 25, 27a, 29
(Underlined Problems are not turned in)

In problems 9, 11, 13, 15, 17 and 19, determine whether the equation is exact. If it is, then solve it.

$$9.) (2xy + 3)dx + (x^2 - 1)dy = 0$$

Computing partial derivatives we obtain:

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}(2xy + 3) = 2x$$
$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}(x^2 - 1) = 2x \quad \Rightarrow \quad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \quad \text{the equation is exact.}$$

Integration yields:

$$F(x, y) = \int M(x, y)dx = \int (2xy + 3)dx = x^2y + 3x + f(y)$$

$$\frac{\partial F}{\partial y} = x^2 + f'(y) = x^2 - 1 \quad \Rightarrow \quad f'(y) = -1 \quad \Rightarrow \quad f(y) = \int -1dy = -y + C \quad \Rightarrow \text{general}$$

solution is

$$F(x, y) = x^2y + 3x - y = C \quad \text{or explicitly,} \quad y = \frac{C-3x}{x^2-1}$$

$$11.) (\cos x \cos y + 2x)dx - (\sin x \sin y + 2y)dy = 0$$

Computing partial derivatives we obtain:

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}(\cos x \cos y + 2x) = -\cos x \sin y,$$
$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}(-(\sin x \sin y + 2y)) = -\cos x \sin y \quad \Rightarrow \quad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \quad \text{the equation is exact.}$$

Integration yields:

$$F(x, y) = \int M(x, y)dx = \int (\cos x \cos y + 2x)dx = \sin x \cos y + x^2 + f(y)$$

$$\frac{\partial F}{\partial y} = -\sin x \sin y + f'(y) = -\sin x \sin y - 2y \quad \Rightarrow \quad f'(y) = -2y \quad \Rightarrow \quad f(y) = \int -2ydy = -y^2$$

solution is

$$F(x, y) = \sin x \cos y + x^2 - y^2 = C$$

$$13.) \left(\frac{t}{y}\right)dy + (1 + \ln y)dt = 0$$

Computing partial derivatives we obtain:

$$\frac{\partial M}{\partial t} = \frac{\partial}{\partial t}\left(\frac{t}{y}\right) = \frac{1}{y}$$
$$\frac{\partial N}{\partial y} = \frac{\partial}{\partial y}(1 + \ln y) = \frac{1}{y} \quad \Rightarrow \quad \frac{\partial M}{\partial t} = \frac{\partial N}{\partial y} \quad \text{the equation is exact.}$$

Integration yields:

$$F(y, t) = \int M(y, t) = \int \left(\frac{t}{y}\right)dy = t \ln|y| + f(t) = t \ln y + f(t)$$

$$\frac{\partial F}{\partial t} = \ln y + f'(t) = 1 + \ln y \quad \Rightarrow \quad f'(t) = 1 \quad \Rightarrow \quad f(t) = \int 1dt = t + c$$

general solution is:

$$F(y, t) = t \ln y + t = C, \text{ or explicitly,} \quad t = \frac{C}{1 + \ln y}$$

$$15.) \cos \theta dr - (r \sin \theta - e^\theta)d\theta = 0$$

Computing partial derivatives we obtain:

$$\frac{\partial M}{\partial \theta} = \frac{\partial}{\partial \theta}(\cos \theta) = -\sin \theta$$

$$\frac{\partial N}{\partial r} = \frac{\partial}{\partial r}(-r \sin \theta - e^\theta) = \sin \theta \quad \Rightarrow \quad \frac{\partial M}{\partial \theta} = \frac{\partial N}{\partial r} \quad \text{the equation is exact.}$$

Integration yields:

$$F(\theta, r) = \int M(\theta, r) = \int (\cos \theta) dr = r \cos \theta + f(\theta)$$

$$\frac{\partial F}{\partial \theta} = -r \sin \theta + f'(\theta) = -r \sin \theta + e^\theta \quad \Rightarrow \quad f'(\theta) = e^\theta \quad \Rightarrow \quad f(\theta) = \int e^\theta d\theta = e^\theta + C$$

general solution is:

$$F(\theta, r) = r \cos \theta + e^\theta = C, \text{ or explicitly, } r = \frac{C - e^\theta}{\cos \theta} = (C - e^\theta) \sec \theta$$

$$17.) \left(\frac{1}{y}\right)dx - \left(3y - \frac{x}{y^2}\right)dy = 0$$

$$M_y = -\frac{1}{y^2} \quad N_x = \frac{1}{y^2} \quad M_y \neq N_x \quad \text{the equation is not exact.}$$

$$19.) \left(2x + \frac{y}{1+x^2y^2}\right)dx - \left(\frac{x}{1+x^2y^2} - 2y\right)dy = 0$$

Computing partial derivatives we obtain:

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y} \left(2x + \frac{y}{1+x^2y^2}\right) = \frac{1+x^2y^2-2x^2y^2}{(1+x^2y^2)^2}$$

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x} \left(\frac{x}{1+x^2y^2} - 2y\right) = \frac{1+x^2y^2-2x^2y^2}{(1+x^2y^2)^2} \quad \Rightarrow \quad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \quad \text{the equation is exact.}$$

Integration yields:

$$F(x, y) = \int M(x, y)dx = \int \left(2x + \frac{y}{1+x^2y^2}\right)dx = x^2 + \arctan xy + f(y)$$

$$\frac{\partial F}{\partial y} = \frac{x}{1+(xy)^2} + f'(y) = \left(\frac{x}{1+x^2y^2} - 2y\right) \quad \Rightarrow \quad f'(y) = -2y \quad \Rightarrow \quad f(y) = \int -2y dy = -y^2 + C$$

solution is

$$F(x, y) = x^2 + \arctan xy - y^2 = C$$

In problems 23 and 25, solve the initial value problem.

$$23.) (e^t y + te^t y)dt + (te^t + 2)dy = 0 \quad y(0) = -1$$

$$M_y = e^t + te^t \quad N_t = te^t + e^t \quad \Rightarrow \quad M_y = N_t \quad \text{the equation is exact.}$$

$$F(y, t) = \int M_y dt = \int N_t dy = \int (te^t + 2)dy = tye^t + 2y + f(t)$$

$$\frac{\partial F}{\partial t} = ye^t + tye^t + f'(t) = e^t y + te^t y \quad \Rightarrow \quad f'(t) = 0 \quad \Rightarrow \quad F(y, t) = tye^t + 2y + c \quad \Rightarrow \quad tye$$

Using the initial condition:

$$y(0) = C/(0 + 2) = -1 \quad \Rightarrow \quad C = -2 \quad \Rightarrow \quad y = -2/(te^t + 2)$$

$$25.) (y^2 \sin x)dx + \left(\frac{1}{x} - \frac{y}{x}\right)dy = 0, y(\pi) = 1$$

Notice this equation is not exact. It is separable.

$$\int x \sin x dx = \int \frac{y-1}{y^2} dy$$

$$-x \cos x + \sin x = \ln y + \frac{1}{y} + C$$

Substituting the initial condition

$$-\pi \cos \pi + \sin \pi = \ln 1 + \frac{1}{1} + C$$

$$C = \pi - 1$$

$$-x \cos x + \sin x = \ln y + \frac{1}{y} + \pi - 1$$

27.) Find the most general function $M(x, y)$ so that the equation is exact.

$$(a) M(x, y)dx + (\sec^2 y - \frac{x}{y})dy = 0$$

We want to find $M(x, y)$ so that for $N(x, y) = (\sec^2 y - \frac{x}{y})$ we have

$$M_y(x, y) = N_x(x, y) = -\frac{1}{y}$$

Therefore, we must integrate this last expression with respect to y :

$$M(x, y) = \int \left(-\frac{1}{y}\right) dy = -\ln|y| + f(x),$$

where $f(x)$ is the constant of integration, a function of only of x .

29.) Consider the equation $(y^2 + 2xy)dx - x^2dy = 0$

(a) Show that the equation is not exact.

$$M_y = 2y + 2x, N_x = -2x \Rightarrow \text{not exact}$$

(b) Show that multiplying both sides of the equation by y^{-2} yields a new equation that is exact

$$y^{-2}(y^2 + 2xy)dx - x^2y^{-2}dy = 0$$

$$\left(1 + \frac{2x}{y}\right)dx - \frac{x^2}{y^2}dy = 0$$

$$M_y = \frac{-2x}{y^2}, N_x = \frac{-2x}{y^2} \Rightarrow \text{exact}$$

(c) Use the solution of the resulting exact equation to solve the original equation.

$$F(x, y) = \int \left(1 + \frac{2x}{y}\right) dx$$

$$= x + \frac{x^2}{y} + g(y)$$

$$F_y = -\frac{x^2}{y^2} + g'(y) = -\frac{x^2}{y^2}$$

$$g'(y) = 0$$

$$g(y) = 0$$

$$x + \frac{x^2}{y} = C$$

$$y = \frac{x^2}{C-x}$$

(d) Where any solutions lost in the process?

Yes, by dividing both sides by y^2 we lost the solution $y = 0$.