Ma 221 Homework Solutions
Due date: January 30, 2009

2.6 p.79 #21, 23, 28;
4.2 p.167 #1, 3, 7, 9, 17, 26, 27 29 (For 27 & 29 you may use the Wronskian.)
(Underlined Problems are not handed in)

2.6 p.79 # 21, 23, 28

For 21, 23 and 28 use the method discussed under "Bernoulli Equations" to solve the problems.

21.)
$\frac{dy}{dx} + \frac{y}{x} = x^2y^2$
This is a Bernoulli Equation with $n = 2$
Let
$u = y^{1-n} = y^{1-2} = y^{-1}$
then $\quad y = u^{-1} \quad \Rightarrow y' = -u^2u'$
and the last equation becomes
$-\frac{1}{u^2}\frac{du}{dx} + \frac{1}{ux} = \frac{x^2}{u^2} \quad \Rightarrow \quad \frac{du}{dx} - \frac{1}{x}u = -x^2$
This is a linear equation with $P(x) = -\frac{1}{x}$
Find $\mu(x) = e^{\int Pdx} = e^{\int(-\frac{1}{x})dx} = e^{(-\ln|x|)} = x^{-1}$
Multiply through by $\mu(x)$ to get
$\frac{1}{x}\frac{du}{dx} - \frac{1}{x^2}u = -x \Rightarrow \quad \frac{d}{dx}(\frac{1}{x}u) = -x$
Integrate to get
$\int \frac{d}{dx}(\frac{1}{x}u) = \int -xdx$
$\frac{1}{x}u = \frac{-x^2}{2} + C_1 \Rightarrow \quad u = -\frac{1}{2}x^3 + C_1x$
Solve explicitly for y
$y = \frac{1}{\frac{x^3}{2}+C_1x} = \frac{2}{Cx-x^3}$
$y = 0$ is also a solution to the original equation. It was lost in the first step when we multiplied by u^2 (also same as dividing by y^2).

23.)
$\frac{dy}{dx} = \frac{2y}{x} - x^2y^2$
or after dividing by y^2 and moving the first term on the right to the left we have
$y^{-2}\frac{dy}{dx} - \frac{2y^{-1}}{x} = -x^2$
Let
$v = y^{-1}$
$v' = -y^{-2}y'$
and the last equation becomes
$v' + 2\frac{v}{x} = x^2$
This is a first order linear equation in v.
The integrating faction for this equation is
$e^{\int \frac{2}{x}dx} = x^2$

Multiplying the DE by this gives
$x^2v' + 2xv = \frac{d}{dx}(x^2v) = x^4$
Thus
$x^2v = \frac{1}{5}x^5 + c_1$
and
$\frac{1}{y} = v = \frac{x^3}{5} + \frac{c_1}{x^2}$
Therefore
$y = \left(\frac{x^3+5c_1}{5x^2}\right)^{-1}$
Letting $C = 5c_1$ we have finally
$y = \left(\frac{5x^2}{x^3+C}\right)$
$y = 0$ is also a solution to the original equation. It was lost in the first step when we divided by y^2.

28.)
$\frac{dy}{dx} + y^3x + y = 0$
Rewrite the equation as
$y' + y = -xy^3$
Multiply both sides by y^{-3} to get
$y^{-3}y' + y^{-2} = -x$
Let
$v = y^{-2}$
$v' = -2y^{-3}y'$
The DE then can be written as
$-\frac{v'}{2} + v = -x$
or
$v' - 2v = 2x$
This is a first order linear equation in v. The integrating factor is
$e^{\int -2dx} = e^{-2x}$
Multiplying the last equation by this leads to
$e^{-2x}v' - 2e^{-2x}v = \frac{d}{dx}(e^{-2x}v) = 2xe^{-2x}$
Integrating gives
$e^{-2x}v = \int(2xe^{-2x})dx = -\frac{1}{2}e^{-2x} - xe^{-2x} + c$
Thus
$\frac{1}{y^2} = -\frac{1}{2} - x + ce^{2x}$
Using SNB to check, we have
$y' + y = -xy^3$, Exact solution is: $\frac{1}{\sqrt{C_1e^{2x}-x-\frac{1}{2}}}$
$C_1 = -4$

4.2 p.167 #1, 3, 7, 9, 17, 26, 27 29
In problems 1, 3 and 9, find a general solution to the given differential equation.

1.)

$y'' + 5y' + 6y = 0$

The auxiliary equation for this problem is $r^2 + 5r + 6 = (r+3)(r+2) = 0$, which has roots $r = -3, r = -2$

Thus $\{e^{-3x}, xe^{-2x}\}$ is a fundamental solution set for this differential equation.

Therefore a general solution is

$y(x) = c_1 e^{-3x} + c_2 xe^{-2x}$

where c_1 and c_2 are arbitrary constants.

3.)

$y'' + 8y' + 16y = 0$

The auxiliary equation for this problem is $r^2 + 8r + 16 = (r+4)(r+4) = 0$, which has the repeated root $r = -4$.

Thus $\{e^{-4x}, xe^{-4x}\}$ is a fundamental solution set for this differential equation.

Therefore a general solution is

$y(x) = c_1 e^{-4x} + c_2 xe^{-4x}$

where c_1 and c_2 are arbitrary constants.

7.)

Find a general solution to the given differntial equation

$2u'' - 7u' - 4u = 0$

The auxiliary equation for this problem is

$2r^2 - 7r - 4 = 0 \Rightarrow (r+4)(2r-1) = 0$ which has roots

$r = -4, \frac{1}{2}$

Thus the general solution to the given diferntial equation is

$y(t) = c_1 e^{-4t} + c_2 e^{\frac{1}{2}t}$

9.)

Find a general solution to the given differntial equation

$y'' - y' - 11y = 0$

The auxiliary equation for this problem is

$r^2 - r - 11 = 0$ which has roots

$r = (1 + - \sqrt{1 + 4 * 11})/2 = (1 + -3\sqrt{5})/2$

Thus the general solution to the given diferntial equation is

$y(t) = c_1 e^{(1+3\sqrt{5})t/2} + c_2 e^{(1-3\sqrt{5})t/2}$

For problem 17, solve the intial value problem.

17.) $z'' - 2z' - 2z = 0$ $\quad z(0) = 0$ $\quad z'(0) = 3$

The auxiliary equation for this problem, $r^2 - 2r - 2 = 0$, has roots $r = 1 + - \sqrt{3}$.

Thus, a general solution is given by $z(t) = c_1 e^{(1+\sqrt{3})t} + c_2 e^{(1-\sqrt{3})t}$

Differentiating, we find that $z'(t) = c_1(1+\sqrt{3})e^{(1+\sqrt{3})t} + c_2(1-\sqrt{3})e^{(1-\sqrt{3})t}$.

Substituting $z(t)$ and $z'(t)$ into the initial condition yields the system

$z(0) = c_1 + c_2 = 0$ $\quad\Rightarrow$

$z'(0) = c_1(1+\sqrt{3}) + c_2(1-\sqrt{3}) = \sqrt{3}(c_1 - c_2) = 3$

$\Rightarrow c_1 = \sqrt{3}/2$

$c_2 = -\sqrt{3}/2$

Thus, the solution satisfying the given initial conditions is

$z(t) = \sqrt{3}/2e^{(1+\sqrt{3})t} - \sqrt{3}/2e^{(1-\sqrt{3})t} = \sqrt{3}/2(e^{(1+\sqrt{3})t} - e^{(1-\sqrt{3})t})$

26.) Boundary Value Problems

Given that every solution to $y'' + y = 0$ is of the form $y(x) = c_1 \cos x + c_2 \sin x$, where c_1 and c_2 are arbitrary constants, show that

(a) there is a unique solution to $y'' + y = 0$ that satisfies the boundary conditions $y(0) = 2$ and $y(\pi/2) = 0$.

(b) there is no solution to $y'' + y = 0$ that satisfies $y(0) = 2$ and $y(\pi) = 0$

(c) there are infinitely many solutions to $y'' + y = 0$ that satisfy $y(0) = 2$ and $y(\pi) = -2$

(a) Plugging $y(0) = 2$ into $y(x)$

$2 = C_1 \cos 0 + C_2 \sin 0$

Therefore

$C_1 = 2$

Plugging $y(\pi/2) = 0$ into $y(x)$

$0 = 2\cos\frac{\pi}{2} + C_2 \sin\frac{\pi}{2}$

Therefore

$C_2 = 0$

The solution with these conditions is

$y(x) = 2\cos x$

(b) Plugging $y(0) = 2$ into $y(x)$

$2 = C_1 \cos 0 + C_2 \sin 0$

Therefore

$C_1 = 2$

Plugging $y(\pi) = 0$ into $y(x)$

$0 = 2\cos\pi + C_2 \sin\pi$

The term with C_2 drops out leaving the equation

$0 = -2$

This is not true, and thus there is no solution that satisfies these boundary conditions.

(c) Plugging $y(0) = 2$ into $y(x)$

$2 = C_1 \cos 0 + C_2 \sin 0$

Therefore

$C_1 = 2$

Plugging $y(\pi) = -2$ into $y(x)$

$-2 = 2\cos\pi + C_2 \sin\pi$

The term with C_2 drops out leaving the equation

$-2 = -2$

This is true, and thus there are infinitely many solutions with these boundary conditions.

27.) Determine whether the functions $y_1(t) = e^{-t}\cos 2t$; $y_2(t) = e^{-t}$ are linearly independent on the interval $(0, 1)$. (For this problem you may use the Wronskian.)

$$W[y_1, y_2] = y_1(t)y_2'(t) - y_1'(t)y_2(t) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

$$W[e^{-t}\cos 2t, e^{-t}] = \begin{vmatrix} e^{-t}\cos 2t & e^{-t} \\ -(\cos 2t)e^{-t} - 2(\sin 2t)e^{-t} & -e^{-t} \end{vmatrix} = 2(\sin 2t)e^{2(-t)} \neq 0$$

Hence these functions are LI.

29.) Determine whether the functions $y_1(t) = te^{2t}$; $y_2(t) = e^{2t}$ are linearly independent on the interval $(0,1)$. (For this problem you may use the Wronskian.)

These two functions are solutions of the DE

$y'' - 4y' + 4y = 0$

since this equation has the characteristic equation

$r^2 - 4r + 4 = (r-2)^2 = 0$

and has the repeated root $r = 2$. Thus we may check to Wronskian. to see if it is zero or not. Now

$$W[te^{2t}, e^{2t}] = \begin{vmatrix} te^{2t} & e^{2t} \\ e^{2t} + 2te^{2t} & 2e^{2t} \end{vmatrix} = e^{4t} \neq 0$$

Hence these functions are LI.